Holography in Flat Space:
Algebraic Geometry and the S-Matrix

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to appear soon
Goal: Discover Dual Theory for S-Matrix

Get $S$ without evolution through spacetime

Evidence it exists: incredible properties of amplitudes, totally obscured by usual insistence on manifest locality.

"WEAK-WEAK" duality $\Rightarrow$ Explicitly see the emergence of spacetime; "decode the hologram" perturbatively!
**Kinematics**

- $p^2 = 0 \Rightarrow p_{\alpha} \tilde{\alpha} = \lambda \tilde{\lambda} \tilde{\tilde{\alpha}}$
- $M(t_1 \tilde{\lambda}_i, t_1 \tilde{\lambda}_i, \tilde{\tilde{\tilde{\tilde{\alpha}}}}_i) = t_i \tilde{t}_i M(\lambda_i \tilde{\lambda}_i, \tilde{\tilde{\alpha}}_i)$
- $e^x$
  
  $M_{\tau \tau} \cdots \tilde{\tilde{\alpha}}_i \cdots + = \frac{<ij>^4}{<123> <23> <n1>}$
- Maximal SUSY
  
  $|\eta\rangle = e^{\langle \tilde{\lambda} \rangle} |\eta\rangle \rightarrow \tilde{\eta} \rightarrow e^{\langle \tilde{\lambda} \rangle} |\eta\rangle$
- $M(\lambda_i \tilde{\lambda}_i, \tilde{\lambda}_i) \ldots \tilde{\tilde{\tilde{\tilde{\alpha}}}_i}$ no discrete labels!
  
  $M_{\alpha} = \sum_{k=0}^{n} M_{n;k} \leftarrow \tilde{\eta}$ charge

- $M_{\alpha} = \sum_{k=0}^{n} M_{n;k}$
  
  $M_{\alpha; k=0,1, n-k, \eta} = 0 \ ; k = 2 \ldots \text{ "MMHV"}$
  
  $k = 3 \ldots \text{ "NMHV"}$
  
  etc.
Twistor Space: Kinematics as simple as possible

\[ M(W) = \int \ldots \mathfrak{d}^{2s} \mathfrak{e} \mathfrak{i}^{\tau} \mathfrak{e}^{\lambda} M(\ldots \mathfrak{e}^{\lambda}, \ldots) \]

\[ M(Z) = \int \ldots \mathfrak{d}^{2s} \mathfrak{e}^{\lambda} M(\ldots \mathfrak{e}^{\lambda}, \ldots) \]

\[ W_{\lambda}^A \equiv \left( \mathfrak{e}^{\lambda} \right)_A^\lambda, \quad Z_{\lambda}^A \equiv \left( \mathfrak{e}^{\lambda} \right)^A_\lambda \]

Conf. Grp: SL(4,R)

\[ M(tW) = t^{2(s-1)} M(W) \]

\[ M(tZ) = t^{2(s-1)} M(Z) \]

SYM: Functions of weight 0 on \( \mathbb{RP}^{3|4} \)

SUGRA: Functions of weight 2 on \( \mathbb{RP}^{3|8} \)
Twistor Space Amplitudes Amazingly Simple

\[ Y_{12}^{3} \frac{\delta^4(\Sigma \lambda \bar{\lambda})}{[13][23]} \]

\[ \text{Gr} : \left( \frac{[12]^3}{[13][23]} \right)^2 \delta^4(\Sigma \lambda \bar{\lambda}) \]

\[ \text{sgn } w_{1}w_{3} \quad \text{sgn } w_{2}w_{3} \quad \text{sgn } [12] \]

\[ |w_{1}w_{3}| |w_{2}w_{3}| |[12]| \]
Symmetries

- Cyclic (perm. for gau)
- Parity

\[ M[\tilde{x}_i, \tilde{y}_i, \tilde{z}_i] \]

\[ \equiv \]

\[ M[\tilde{x}_{i+1}, \tilde{y}_{i+1}, \tilde{z}_{i+1}] \]

\[ \int_0^\pi e^{i \tilde{z}_i} M(\tilde{x}_i, \tilde{y}_i, \tilde{z}_i) \]

\[ \equiv \]

\[ M[\tilde{x}_i, \tilde{y}_i, \tilde{z}_i \rightarrow \tilde{y}_i] \]
Singularity Structure: Imprint of Locality

. Trees

. One Loop

\[
\sum_{\text{boxes}} \left( \begin{array}{c}
\text{+ subleading,} \\
\text{vanish = } \text{maximal SUST}
\end{array} \right)
\]
\[ \text{IR Limits} \]

\[
M^{1\text{-loop}}_{\text{IR}} = \left( \sum_i - \frac{1}{\epsilon^2} \left( \delta_{ji} \mu \right) \right) M^{\text{tree}}_n
\]

\[ \text{[many eqns...]} \]
\[ \text{IR eqns @ l-loop} \sum_{l,R} \xrightarrow{\text{BCF}} \sum_{l,R} \frac{1}{p_l^2} \]
BCFW $6 \to 5$ NMHV

\[
\frac{\langle 46 \rangle^4 \langle 13 \rangle^4}{[a_2][a_2][a_3]} \frac{1}{(p_1 + p_2 + p_3)^2} \frac{1}{\langle 6 l 5 + 4 l 3 \rangle} \frac{1}{\langle 4 l 5 + 6 l 4 \rangle}
\times \{i \to i+2\} + \{i \to i+4\}
\]

"Spurious Poles: Don't occur in local theories!"
Remarkable 6-term Id

\[
\frac{<46^4[13]^4}{[43][23][45][56]} \times \frac{1}{(p_a+p_b+p_c)^2} + \{i \rightarrow i+2\} + \{i \rightarrow i+4\}
\]

\[
\frac{<31(2+f^4)6^4}{[23][34][56][67]} \times \frac{1}{(p_5+p_6+p_7)^2} + \{i \rightarrow i+2\} + \{i \rightarrow i+4\}
\]

\[
\frac{1}{<615+413]} \quad \frac{1}{<415+613]} \quad \frac{1}{<16+514]} \quad \frac{1}{<516+12]}
\]

Guarantees

\{ Parity, Cyclic, No Spurious Poles \}

7-pi
8-pi
12 terms
20 terms
40 terms
SOME POWERFUL
MATHEMATICAL
STRUCTURE
IS AT WORK!
The Conjectured Duality

\[ Q_{n,k} = \int \prod_{j=1}^{n} \frac{d^{nx_k} C_{\alpha \beta}}{(2 \cdot \cdot \cdot k) \cdot \cdot (n \cdot \cdot \cdot k-1)} \prod_{\alpha=1}^{k} \sum_{\mu} \delta_{\mu \mu} [ C_{\alpha \beta} ] \]
. Claim: after we make this sharply defined trivial to back to momentum space — multi-dimensional contour integral.

. Residues compute 1-loop leading singularities (+ hence all 1-loop amps) in $\mathcal{N}=4$ SYM!

. (Includes all tree amps too by BCF logic)
- \( C_\alpha = \begin{pmatrix} c_{11} & \cdots & c_{1\eta} \\ \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{kn} \end{pmatrix} \) \( \rightarrow \) \( k \)-plane in \( n \)-space \( \gamma \in G(k,n) \)

- "Gauge Symmetry" \( C_{\alpha} \rightarrow C_{\beta} L^\alpha \), any \( k \times k \) \( L \)

- "Gauge Fix": columns I to some basis e.g.

\[
C = \begin{pmatrix} 1 & 0 & \cdots & c_{17} \\ 0 & 1 & \cdots & c_{17} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}
\]  

or

\[
C = \begin{pmatrix} 1 & c_{21} & c_{41} & c_{61} & c_{71} \\ 0 & c_{23} & c_{43} & c_{63} & c_{73} \\ 0 & c_{25} & c_{45} & c_{65} & c_{75} \end{pmatrix}
\]

Can choose any G-Fing we like. [Different Charts on Grassmannian]

Leaves us with \( k \) \((n-k)\) variables.
Obvious mapping between \( k \)-plane and \( \perp (n-k) \) plane.

Given e.g.

\[
C = \begin{pmatrix} 1_k & \mathbf{c} \end{pmatrix}, \text{ define } \star C = \begin{pmatrix} 1_{n-k} & -\mathbf{c}^T \end{pmatrix}
\]

Symmetry \( k \leftrightarrow (n-k) \) is just parity.
After GFing, trivial to go back to momentum space:

\[
Q_{n,k} = \int \frac{d^{(k-k)_k}}{c^{(1...k)...(n-k-1)}} \delta^2 [C^2] \delta^2 [\ast C^2]
\]

\[
S^4 [\ast C^2]
\]

*Parity Manifest!*
Note: for gluon amplitudes, convenient to gauge fix columns I corresponding to particles of negative helicity—then intgd is always the same, it's just form of $C_i$ that changes.

e.g. $^{123456}_7$ $^{1234567}_8$

$$C = \begin{pmatrix} 1 & 0 & c_{31} & 0 & c_{51} & c_{61} \\ 0 & 1 & c_{32} & 0 & c_{52} & c_{62} \\ 0 & 0 & c_{34} & 1 & c_{54} & c_{64} \end{pmatrix} \quad C = \begin{pmatrix} 1 & c_{21} & 0 & c_{41} & 0 & c_{61} \\ 0 & c_{22} & 1 & c_{42} & 0 & c_{62} \\ 0 & c_{25} & 0 & c_{45} & 1 & c_{65} \end{pmatrix}$$

Diff. helicities: integrate same fn. on different charts of Grassmannian!
The condition $CA = 0$, $x \tilde{C} \tilde{x} = 0$

$\Leftrightarrow \lambda_i - c_i \tilde{\lambda}_i = 0$, $\tilde{\lambda}_i + c_i \tilde{\lambda}_i = 0$

Can only be satisfied if mom. conserved

$\lambda_i \tilde{\lambda}_i + \lambda_i \tilde{\lambda}_i = 0$!
Geometrically: \( \vec{a}, \vec{b}, \vec{c} \) in vectors.

\[ \vec{a} \text{ 2-plane} \]

\[ \vec{c} \text{ k-plane, orthog. to } \vec{a}, \text{ must contain } \vec{b} \]

\[ \Rightarrow \vec{a} \cdot \vec{c} = 0 = \text{ mom conservation!} \]
So, we are left with \( k(n-k)-(2n-4) \)
\[
= (k-2)(n-k-2) \quad \text{parameters: signs of}
\]
\[
\lambda_i - c_i I(\gamma_A) I I = 0, \quad \tilde{\lambda}_I + c_I I(\gamma_A) = 0
\]

\[
c_i I(\gamma_A) = c_i I + d_i I A \gamma^A, \quad A = 1, \ldots, (k-2)(n-k-2)
\]

[one to one correspondence with \( G(k-2, n-4) \)]
Note: no such plane exists for $k=0,1,n-1,n!$
That's why those amps vanish.

$k=2$, $n-2$ \[MHV + \overline{MHV}\], plane fixed but we get right answer.

Otherwise

(k-2) \[n-k-2\] directions
k-plane can rotate in!
Factoring out $S^q(\Sigma \eta a)$, we have

$$Q_{n,k}(\lambda, i) = \mathcal{I}(\lambda, i) \int \frac{d \lambda}{(12 \cdots k)(2)(3) \cdots (n1 \cdots k-1)(2)}$$

At this point everything can be fully complexified.

Easy to prove: each minor is of degree

$$\min \{ k-2, n-k-2 \} \ in \ the \ \mathcal{I}'s. \ \{ k=3, \ all \ linear \}$$
First interesting case: 6 pt NMHV ($n=6, k=3$).

Look at $1^+2^-3^+4^-5^+6^-$

$$C = \begin{pmatrix}
\ast & 1 & \ast & \ast & \ast \\
\ast & \ast & 1 & \ast & \ast \\
\ast & \ast & \ast & \ast & 1 \\
\ast & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

$$\lambda_i - c_{i1} \lambda_1 = 0, \quad \tilde{\lambda}_i + c_{i1} \tilde{\lambda}_1 = 0$$

$$\Rightarrow \quad c_{i1}(\mathcal{Z}) = c_{i1} + \varepsilon_{ijk} \varepsilon_{1jk} \langle jk \rangle [ijk]$$

Jacob. $\delta^i(\Sigma p_2) \rightarrow \mathcal{J}(\lambda, \tilde{\lambda}) = 1$. 
So,

\[ Q_{b,3} = \int d\tau \frac{1}{(123)(234)\cdots(612)(\tau)} \]

Each factor linear in \( \tau \)
- residues: BCFW terms
- residues: $P[BCFW]$ terms
- Cauchy: $BCFW = P[BCFW] = \text{Remarkable 6-term identity!}$
Spurious Poles

Contour can be deformed away from singularity

Physical Poles

Can't deform contour to avoid singularity

LOCALITY $\leftrightarrow$ CONTOUR DEFORMATION
For all other cases we have more than 1 complex variable. What is a residue?

\[ f(z) = \frac{g(z)}{A_1(z) \cdots A_n(z)} \quad z = (z_1, \ldots, z_n) \]

\[ \text{res} \ f(z*) \quad A_1(z*) = \cdots = A_n(z*) = 0 \]

\[ = \frac{g(z*)}{\det \frac{\partial A_i}{\partial z_j} \mid_{z^*}} \]
Also, higher-dim gen of Cauchy’s thm:

\[
f(z) = \frac{g(z)}{A_1(z) \cdots A_n(z)}
\]

[Note: a given \( f(z) \) can be written in this form in many ways]

Then

\[
\sum_{\substack{z = z_1 \cdots = z_n \neq 0 \text{ } \forall i \in \{1, \ldots, n\}}} \text{res } f(z) = 0 \quad \{ \text{if deg } g \text{ is small enough} \}
\]
First new case: \( m = 7, \ k = 3 \) (7 pt NMHV). Look e.g. @ 1\textsuperscript{+}2\textsuperscript{-}3\textsuperscript{+}4\textsuperscript{-}5\textsuperscript{+}6\textsuperscript{-}7\textsuperscript{-}.

\[
\mathcal{Q}_{7,3} = \int \frac{dz_1 \, dz_2}{(123) \ldots (712)}
\]

There are \( \binom{7}{2} = 21 \) residues.
They are computed just by solving linear equations, but can get interesting!

e.g. \((123)(456) = \left( \frac{\langle 7| (2+4) |3\rangle <5\rangle + \langle 7|6\rangle <5\rangle <3\rangle <4\rangle}{\langle 2\rangle <3\rangle <4\rangle <5\rangle <6\rangle <7\rangle} \right)^4 \\
\times \left( \frac{1}{\langle 11| (2+3) |4\rangle <7\rangle <5\rangle <6\rangle <7\rangle} \right)

+ 20 other horrible guys!
Miraculously, these 21 residues exactly match the 21 objects first found in 04 by Zvi, Lance et al. in the 1-loop $\mathcal{N}=4$ amp.

\[
  \leftrightarrow \quad (123)(671)
\]

\[
  \leftrightarrow \quad (123) \left[ (345) + (567) + (742) \right] + (456)(567)
\]
Now, the Global Residue Thm implies many identities. [Actually overkill in this case since all factors linear – just repeated use of Cauchy gives same thing].

\[ f(z_1, z_2) = \frac{1}{(1) \left[ (2)(3) \cdots (7) \right]} \Rightarrow (1)(2) + \cdots + (1)(7) = 0 + \text{cyclic exhaust all residue id.} \]
These "basic" identities have a direct physical interpretation: IR equations!

E.g. look at \( \log \left( \left( q_1 + q_2 + q_3 \right)^2 \right) \) term in 1-loop IR

\[
eq n = (1)(2) + (1)(3) + \ldots + (1)(7) = 0.
\]

**LOCALITY** $\leftrightarrow$ **CONTOUR DEFORM**.
Take

\[ f(z_1, z_2) = \frac{1}{(1)(2)(3)(4)(5)(6)(7)} \overbrace{A_1}^{A_1} \overbrace{A_2}^{A_2} \]

4 × 3 = 12 term identity which guarantees cyclic.

+ absence of spurious poles for 7 pt tree amp!

( Can actually show there is a unique such object \( \rightarrow \) build local 7pt amp).
. We know the explicit map between residues and boxes for all NMHV amplitudes:

$$\begin{array}{c}
\frac{1}{(n-1 \, n \, 4)} \quad (123) \prod_{i=2}^{i=s} (i+1) \prod_{i=t}^{i=s} (i+1) \\
\left(\frac{n-1 \, n \, 4}{(n-1 \, n \, 4)} + (234)\right) \prod_{i=3}^{i=s} (i+1) \prod_{i=1}^{i=t} (i+1) + (i+2)
\end{array}$$
All $G_1, R > 0$ Thm ident. linear comb. of

$$\sum_i (i_1)(i_2) \cdots (i_{n-6}) (j) = 0$$

**BUNCH OF 6 term id.**
Simplest IR Finite 1-loop:

\[ 8 \; p^+ \; N^2 \text{MHV} \]
\[ \sum_{\ell^+ \ell^-} \frac{\langle 23 \rangle^3 [67 \rangle^3 [41 \langle l - p_{23} \rangle^3 \rangle <1\rangle^3}{\langle 15 \rangle \langle 18 \rangle \langle 11 \rangle \langle 31 \rangle \langle l - p_{23} \rangle^4} \times \frac{1}{[4 \langle p_{23} \rangle^2 \langle e \rangle \langle p_{16} \rangle] \langle 51 \langle l - p_{23} \rangle \rangle \langle 0 \rangle \langle l - p_{23} \rangle \langle l - p_{23} \rangle^4} \]

where

\[ \ell^2 = (l - K_1)^2 = (l - K_1 - K_2)^2 = (l + K_4)^2, \quad \text{quad. eqn with} \]

\[ \Delta = 1 - 2(p_1 + p_2) + (q_1 - p_2)^2; \quad p_1 = \frac{K_1^2 K_2^2}{K_1^2 K_2^2}, \quad p_2 = \frac{K_1^2 K_4^2}{K_1^2 K_2^2} \]
Very Non-trivial Check!

\[ \text{res} \left[ \frac{1}{(1234) \cdots (8123)} \right] \]

where \((1234) = (4567) = (6781) = (8123) = 0\).

Actually these are quadratic in 4 \(z\)'s, get 2 solns, corresponding to \(\ell^\pm\)!
The object we have found seems to unify all 1-loop leading singularities. Very natural and beautiful mathematical structure — intersection theory + Schubert Calculus — seems to lie at the heart of tree + loop gluon scattering amplitudes!
For ex: $n = 10$, $k = 5$

\[
\frac{1}{(12345) \ldots (101234)} \leftrightarrow 9 \times 2 \text{'s}
\]

cubic in 2's

residue: simultaneous soln of 9 cubics in 9 variables...

help!
General answer: \[ \mathcal{D} = (k-2)(n-k-2) \times \dim \mathcal{G}(k-3, n-4) \]

# self-int of 

\[ \mathcal{D} \]

elem. Schubert cycle

\[ \| \]

# of appearances of \( \begin{array}{c} \overset{k-2}{\text{}} \\ \overset{n-k-2}{\text{}} \end{array} \) in \( \mathcal{D} \)

\[ = \frac{1 \cdot 2! \cdots (k-3)! \cdot \mathcal{D}!}{(n-k-2)! \cdots (n-5)!} \]

\[ = 42 \text{ for } k = 5 \]

( Works! )
But perhaps our physical interpretation has something to offer mathematicians?

Naively, expect solns of cubics to involve \( \sqrt[3]{\cdot} \)’s.

But in \( \sqrt[4]{\cdot} \)’s appear! \[ 4D \text{ Locality again!} \]
So we predict that, no matter how high $k, n \geq 1$ get, that if the external momenta are in $\mathbb{Q}$, the solns are in $\mathbb{Q} + \sqrt[n]{\mathbb{Q}}$, no higher roots! 

We've checked that this is correct for $k = 5, \ n = 10$. 
All of these checks reveal the power of a weak-weak duality. And we are beginning to see the sorts of structures that allow local spacetime physics to emerge holographically.
Outlook

- Quantum
- S-Matrix
- Alg. Geometry
- Twistor

The dual theory clearly exists.

What is the physics?
\[ Q_{n,k} = \int dX^{(j)}_a dY^{(j)}_\beta dZ^a dC_{\alpha \delta} e^{iS} \]

\[ S = X^{(j)}_a Y^{(j)}_\beta C_{j+a-1,\beta} + \lambda a C_{\alpha \delta} Z^a \]
\[ N = 8 \text{ Will Be Much More Interesting!} \]