Scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills

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Based on work in collaboration with
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Motivation and outline

✓ Why $\mathcal{N} = 4$ super Yang-Mills?
  × learn something about pure Yang-Mills
    e.g. tree-level gluon scattering amplitudes

  × spectrum of anomalous dimensions
    expected integrability
    compute at weak and strong coupling (AdS/CFT)

  × integrability for other quantities in $\mathcal{N} = 4$ SYM?

✓ Questions we want to ask:
  × can we compute tree-level amplitudes for an arbitrary number of gluons?
  × what are the symmetry properties of the amplitudes?
First part - computing tree-level amplitudes
Tree-level scattering amplitudes

- maximally helicity violating (MHV) amplitudes

\[ A_{n}^{\text{MHV}}(i, j) = \delta^{4}(p) \frac{\langle i j \rangle^{4}}{\langle 12 \rangle \ldots \langle n \, 1 \rangle} \]

\[ p_{i}^{\alpha \dot{\alpha}} = \lambda_{i}^{\alpha} \tilde{\lambda}_{i}^{\dot{\alpha}} , \quad \langle i \, j \rangle = \lambda_{i}^{\alpha} \lambda_{j}^{\beta} \epsilon_{\alpha \beta} \]

- MHV amplitudes in \( \mathcal{N} = 4 \) on-shell superspace

\[ \Phi = g^{+} + \eta^{A} f_{A} + \ldots + 1/4! \epsilon_{ABCD} \eta^{A} \eta^{B} \eta^{C} \eta^{D} g^{-} , \quad q_{i}^{\alpha A} = \lambda_{i}^{\alpha} \eta_{i}^{A} \]

\[ A_{n}^{\text{MHV}} = \delta^{4}(p) \delta^{8}(q) \frac{1}{\langle 12 \rangle \ldots \langle n \, 1 \rangle} \]

\( N^{k} \text{MHV}_n \) amplitudes
On-shell recursion relations

✓ on-shell recursion relations

\[ A = \sum A_L \frac{1}{P^2} A_R \]

× follow from analytic properties of the amplitudes (Feynman diagrams)

× \( n \)-point amplitudes are obtained recursively from lower-point amplitudes

× all amplitudes are on-shell

✓ \( \mathcal{N} = 4 \) supersymmetric version

[Britto, Cachazo, Feng + Witten, 2004]

[Arkani-Hamed, Cachazo, Kaplan 2008]
All tree-level amplitudes in $\mathcal{N} = 4$ SYM

Supersymmetry implies

$$A_n = \delta^4(p)\delta^8(q) \frac{1}{\langle 12 \ldots n \rangle} \mathcal{P}_n$$

$$\mathcal{P}^\text{MHV}_n = 1 \quad [\text{Nair 1988}]$$
All tree-level amplitudes in $\mathcal{N} = 4$ SYM

Supersymmetry implies

$$\mathcal{A}_n = \delta^4(p)\delta^8(q)\frac{1}{\langle 12 \rangle \cdots \langle n \rangle} \mathcal{P}_n$$

1. $\mathcal{P}^{\text{MHV}}_n = 1$ [Nair 1988]
2. $\mathcal{P}^{\text{NMHV}}_n = \sum_{2 \leq i, j \leq n-1} R_{n;i,j}$ [Drummond, J.M.H., Korchemsky, Sokatchev 2008]

with

$$R_{n;i,j} = \frac{\langle i \ i - 1 \rangle \langle j \ j - 1 \rangle \delta^4(\Xi_{n;ij})}{x_{ij}^2 < n | x_{ni} x_{ij} | j > < n | x_{ni} x_{ij} | j - 1 > < n | x_{nj} x_{ji} | i > < n | x_{nj} x_{ji} | i - 1 >}$$

where

$$\Xi_{n;ij} = < n | x_{ni} x_{ij} | \theta_{jn} > + < n | x_{nj} x_{ji} | \theta_{in} >$$

and

$$p_i = x_i - x_{i+1}, \quad q_i = \theta_i - \theta_{i+1}$$
All tree-level amplitudes in $\mathcal{N} = 4$ SYM

supersymmetry implies

$$A_n = \delta^4(p)\delta^8(q) \frac{1}{\langle 12 \rangle \ldots \langle n 1 \rangle} \mathcal{P}_n$$

✓ $\mathcal{P}^{\text{MHV}}_n = 1$ [Nair 1988]

✓ $\mathcal{P}^{\text{NMHV}}_n = \sum_{2 \leq i, j \leq n-1} R_{n;i,j}$ [Drummond, J.M.H., Korchemsky, Sokatchev 2008]

✓ generic case [Drummond, J.M.H. 2008]

$$\mathcal{P}^{N^k\text{MHV}}_n = \sum \cdots \sum_{2k\text{-fold nested sums}} R \times \cdots \times R_{k\text{-fold product}}$$
Second part - symmetries
Expected symmetries of the amplitudes

- \textit{psu}(2, 2|4) superconformal generators in on-shell superspace \cite{Witten 2003}

\[
[J_a, J_b] = f_{ab}^c J_c, \quad J_a = \sum_{i=1}^{n} J_{ia}
\]

\[
p^{\dot{\alpha} \alpha} = \sum_{i} \tilde{\lambda}_{i}^{\dot{\alpha}} \lambda_{i}^{\alpha},
\]

\[
d = \sum_{i} \left[ \frac{1}{2} \lambda_{i}^{\alpha} \frac{\partial}{\partial \lambda_{i}^{\alpha}} + \frac{1}{2} \tilde{\lambda}_{i}^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{i}^{\dot{\alpha}}} + 1 \right],
\]

- invariance under \textit{psu}(2, 2|4)

\[
\{p, k, m, \bar{m}, d, c, r, q, \bar{q}, s, \bar{s}\} A = 0
\]
Hints for a new symmetry

✓ loop corrections to four-gluon (MHV) amplitude [Bern, Dixon, Smirnov 2005]

✓ hints for a new symmetry [Drummond, J.M.H., Smirnov, Sokatchev 2006]
introduce dual variables

\[ x_i - x_{i+1} = p_i \]

conformal symmetry in dual space! (broken by infrared divergences)

✓ same variables also appeared in AdS dual [Alday, Maldacena 2007]


Dual superconformal symmetry

✓ amplitudes have additional symmetries!

[Drummond, J.M.H., Korchemsky, Sokatchev ’08]

\[x_i^{\alpha \dot{\alpha}} - x_{i+1}^{\alpha \dot{\alpha}} = p_i^{\alpha \dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}}\]

✓ dual conformal generator

\[K_{\alpha \dot{\alpha}} = \sum_i \left[x_i^{\beta} x_i^{\beta} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} \right]\]
Dual superconformal symmetry

✓ amplitudes have additional symmetries!

[Drummond, J.M.H., Korchemsky, Sokatchev ’08]

\[ x_i^{\alpha \dot{\alpha}} - x_{i+1}^{\alpha \dot{\alpha}} = p_i^{\alpha \dot{\alpha}} = \lambda_i^{\alpha \tilde{\lambda}_i^{\dot{\alpha}}} \]

✓ dual conformal generator

\[ K_{\alpha \dot{\alpha}} = \sum_i \left[ x_i^{\alpha \beta} x_i^{\dot{\alpha} \beta} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_{i+1}^{\alpha \beta} \lambda_i^{\alpha \dot{\beta}} \frac{\partial}{\partial \lambda_i^{\beta \dot{\beta}}} \right] \]
Dual superconformal symmetry

- amplitudes have additional symmetries!
  
  \[ x_i^{\alpha \dot{\alpha}} - x_{i+1}^{\alpha \dot{\alpha}} = p_i^{\alpha \dot{\alpha}} = \lambda_i^{\alpha} \tilde{\lambda}_i^{\dot{\alpha}} \]

- dual conformal generator
  
  \[ K_{\alpha \dot{\alpha}} = \sum_i \left[ x_i^{\alpha \dot{\beta}} x_i^{\dot{\alpha} \beta} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_i^{\alpha \dot{\beta}} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\alpha \dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\beta}} \right] \]

- supersymmetric generalisation
  
  \[ \theta_i^A \alpha - \theta_{i+1}^A \alpha = q_i^A \alpha = \eta_i^A \lambda_i^{\alpha} \]

- final expression for dual conformal generator
  
  \[ K^{\alpha \dot{\alpha}} = \sum_i \left[ x_i^{\alpha \dot{\beta}} x_i^{\dot{\alpha} \beta} \frac{\partial}{\partial x_i^{\beta \dot{\beta}}} + x_i^{\alpha \dot{\beta}} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_{i+1}^{\alpha \dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\beta}} + x_i^{\dot{\alpha} \beta} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} + \tilde{\lambda}_i^{\dot{\alpha}} \theta_{i+1}^{\alpha B} \frac{\partial}{\partial \eta_i^{\beta B}} \right] \]
Dual superconformal symmetry

✓ act on variables \( \{ \lambda_i, \tilde{\lambda}_i, \eta_i, x_i, \theta_i \} \)

✓ constraints \( x_i - x_{i+1} = \lambda_i \tilde{\lambda}_i, \quad \theta_i - \theta_{i+1} = \eta_i \lambda_i \)

\[
\begin{align*}
P_{\alpha\dot{\alpha}} &= \sum_i \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}, \\
Q_{\alpha A} &= \sum_i \frac{\partial}{\partial \theta_i^{\alpha A}}, \\
\bar{Q}_{\dot{\alpha}} &= \sum_i [\theta_i^{\alpha A} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} + \eta_i^A \frac{\partial}{\partial \eta_i^{\dot{\alpha}}}],
\end{align*}
\]

\[
K^{\alpha\dot{\alpha}} = \sum_i \left[ x_i^{\alpha\beta} x_i^{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial x_i^{\beta\dot{\beta}}} + x_i^{\dot{\alpha}\dot{\beta}} \lambda_i^{\alpha} \frac{\partial}{\partial \lambda_i^{\beta}} + x_i^{\alpha\dot{\beta}} \tilde{\lambda}_i^{\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{\beta}}} + x_i^{\dot{\alpha}\dot{\beta}} \theta_i^{\alpha B} \frac{\partial}{\partial \theta_i^{\beta B}} + \tilde{\lambda}_i^{\dot{\alpha}} \eta_i^{\dot{\alpha}} + \lambda_i^{\alpha} \right]
\]

Similarly, \( M_{\alpha\beta}, M_{\dot{\alpha}\dot{\beta}}, R_{AB}, D, C, S_{\alpha A}, S_{\dot{\alpha}A} \).

Tree amplitudes are covariant under dual superconformal transformations

\[
K^{\alpha\dot{\alpha}} \mathcal{A}_n = - \sum_i x_i^{\alpha\dot{\alpha}} \mathcal{A}_n
\]

[Drummond, J.M.H., Korchemsky, Sokatchev ’08]

[Brandhuber, Heslop, Travaglini ’08], [Drummond, J.M.H. ’08]
Conventional and dual superconformal symmetry

[Drummond, J.M.H., Korchemsky, Sokatchev ’08]
\[
p \quad K
\]
\[
q \quad \bar{q} = \bar{S} \quad S
\]
\[
s \quad \bar{s} = \bar{Q} \quad Q
\]
\[
k \quad P
\]

✓ also observed in the AdS dual (fermionic T-duality)
   [Maldacena, Berkovits ’08] [Beisert, Ricci, Tseytlin, Wolf ’08]

✓ closure of the algebra?
Commuting the two algebras

[Drummond, J.M.H., Plefka 2009]

✓ technical steps
  × reformulate covariance as an invariance

\[(K_{\dot{\alpha}} + \sum_i x_i \dot{\alpha}) A_n = 0\]

× remove all \(x, \theta\) dependence (use \(P\) and \(Q\) to set \(x_1 = 0\) and \(\theta_1 = 0\))

× subtract ‘trivial’ terms like \(pd\)

✓ we find \(K'_{\dot{\alpha}} A_n = 0\) with

\[K_{\dot{\alpha}} \rightarrow K'_{\dot{\alpha}} = \sum_{i>j} \left[ \left( m^{\gamma}_{i\dot{\alpha}} \delta_{\dot{\gamma}}^i + \bar{m}^{\gamma}_{i\dot{\alpha}} \delta_{\dot{\gamma}}^i - d_i \delta_{\dot{\alpha}}^\gamma \delta_{\dot{\gamma}}^i \right) p_{j\gamma} q_i + \bar{q}_{i\dot{\alpha}} c q_j^c - (i \leftrightarrow j) \right] \]

✓ notice

\[K'_{\dot{\alpha}} = f_{(p_{\dot{\alpha}}}^{bc} \sum_{i<j} J_{ib} J_{jc} \]

where

\[[J_a, J_b] = f_{ab}^c J_c, \quad J_a = \sum_{i=1}^n J_{ia} \]
Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills

- superconformal symmetry $psu(2,2|4)$
  
  $psu(2,2|4)$ algebra:
  \[
  [J_a, J_b] = f_{ab}{}^c J_c, \quad J_a = \sum_{i=1}^{n} J_{ia}
  \]

- dual superconformal symmetry
  
  [Drummond, J.M.H., Korchemsky, Sokatchev 2008]

- closure of algebra gives Yangian $Y(psu(2,2|4))$
  
  [Drummond, J.M.H., Plefka 2009]

- level-one Yangian generators
  
  \[
  Q_a = f_a{}^{cb} \sum_{1 \leq i < j \leq n} J_{ib} J_{jc}, \quad [Q_a, J_b] = f_{ab}{}^c Q_c
  \]

- interesting observation: form of $Q_a$ consistent with cyclicity for certain superalgebras only

- Yangian symmetry of scattering amplitudes also expected in string theory
  
  [Beisert 2009]
Summary

- solved problem of computing tree-level amplitudes in $\mathcal{N} = 4$ SYM applications:
  - easy to extract gluon amplitudes
  - useful for loop computations via generalised unitarity
  - same technique of solving the recursion relations applicable to $\mathcal{N} = 8$ supergravity

- new symmetries of tree-level amplitudes
  - conventional and dual superconformal symmetry combine to Yangian
  - sign of integrability?
Outlook and open questions

✓ divergences appear at loop level

✓ do loop-level amplitudes also have a Yangian symmetry?

✗ breaking of dual conformal symmetry controlled by Ward identity

what about conventional superconformal symmetry?

✗ connection to Wilson loops? Wilson loops in dual chiral superspace

\((x^{\mu}_{i}, \theta^{A}_{i})\)?

✗ insights from AdS/CFT?