Gravity at a Lifshitz Point

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Roma, June 25, Strings 2009

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based on:

arXiv:0811.2217, arXiv:0812.4287, arXiv:0901.3775, arXiv:0902.3657,

plus papers in preparation with Charles Melby-Thompson, and work in progress with CMT and Kevin Grosvenor.

related work by: Bertoldi, Blas, Botta-Cantcheff, Brandenberger, Burrington, Cai, Calgani, Cao, Carvalho, Charmousis, Chen, Cheng, Colgain, Comelli, Das, Dhar, Gao, Germani, Ghodsi, Ghosh, Grandi, Ha, Hatefi, Hu, Huang, Iengo, Izawa, Jing, Kalyana Rama, Kehagias, Kim, Kiritsis, Klusoň, Kofinas, Konoplya, Leite, Li, Liu, Liberati, Lu, Ma, Maccione, Mandal, Mann, Mattingly, Mei, Minamitsui, Matsuda, Modesto, Moffat, Mukohyama, Myung, Nakayama, Nastase, Nesti, Nicolic, Nishioka, Niz, Nojiri, Taylor, Odintsov, Ohta, Orlando, Padilla, Pal, Pang, Pechan, Peet, Pi, Piao, Pilo, Pope, Premont-Schwarz, Reffert, Renner, Riotto, Takahashi, Tang, Saffin, Sakamoto, Saridakis, Sfetsos, Shu, Soda, Sotiriou, Sturla, Sun, Visser, Volovich, Volovik, Wadia, Wang, Weinfurtner, Wen, White, Wu, Yavartanoo, Yukoyama, Zhang, . . .

Central idea

Combine gravity with the concept of anisotropic scaling.

In a spacetime with coordinates $(t, \mathbf{x}) \equiv (t, x^i)$, $i = 1, \ldots D$, consider

$$\mathbf{x} \to b\mathbf{x},$$
$$t \to b^z t.$$

Here z is the dynamical critical exponent.

In condensed matter, many values of z are possible; not just integers (1, 2, . . .) but also fractions ($z = 3/2$ for KPZ surface growth in $1 + 1$ dimensions, . . .).

Example: Lifts of static critical systems (Euclidean QFTs) to dynamical critical phenomena.

Why?

String theory is a beautiful theory of quantum gravity, but it appears both "too large" and "too small."

Is all of string theory required for QG, or could QG be more like Yang-Mills?

Lessons from string theory:

Quantum mechanics is absolute, but GR undergoes corrections.

Lorentz symmetry unlikely to be fundamental, if space is emergent!

Motivation from string theory:

Strings and gravity out of equilibrium (methods will be those of condensed matter, where nonequilibrium situations are often studied.

Possible applications

expected in four areas:

(i) Gravity in our Universe of $3 + 1$ dimensions

(ii) Gravity on worldvolumes of branes

(iii) Gravity duals of field theories in AdS/CFT

(iv) Mathematical applications

Recall: What makes strings special?

Consider branes fluctuating in ambient space.
Four "coincidences" pick out $1+1$ dimensions.

(1) (lower) critical dimension of scalar fields,

$$W = \frac{1}{2}\int d^D\mathbf{x}\,(\partial\phi)^2, \qquad [\phi] = \frac{D-2}{2}.$$

(2) critical dimension of gravity, $[G_N] = 0$.

⋮

Coincidence (1) implies an infinite number of classically marginal couplings in the theory. (Such systems are rare; cf. Fermi liquids.)

In the quantum theory, marginality implies Einstein's equations!

Review of Lifshitz scalar field theory

A condensed-matter trick to promote the static universality class given by W to a dynamical one: Promote $\phi(\mathbf{x})$ to $\phi(t, \mathbf{x})$, write

$$S = \frac{1}{2} \int dt\, d^D\mathbf{x} \left\{ \dot{\phi}^2 - \left(\frac{1}{2} \Delta \phi \right)^2 \right\}$$

$\Psi_0[\phi(\mathbf{x})] = \exp\{-W/2\}$ is a groundstate wavefunction, because

$$\left(\frac{\delta}{\delta \phi} - \frac{1}{2} \Delta \phi \right) \Psi_0 = 0.$$

Critical dimension has shifted:
$[\phi] = \frac{D-2}{2}$, ϕ is dimensionless in $2+1$ dimensions.

Relevant deformations and RG flows

The Lifshitz scalar can be deformed by relevant terms:

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\, \left\{ \dot{\phi}^2 - \left(\frac{1}{2}\Delta\phi\right)^2 - \mu^2(\partial_i\phi)^2 + m^4\phi^2 \right\}$$

Properties of the IR flow:

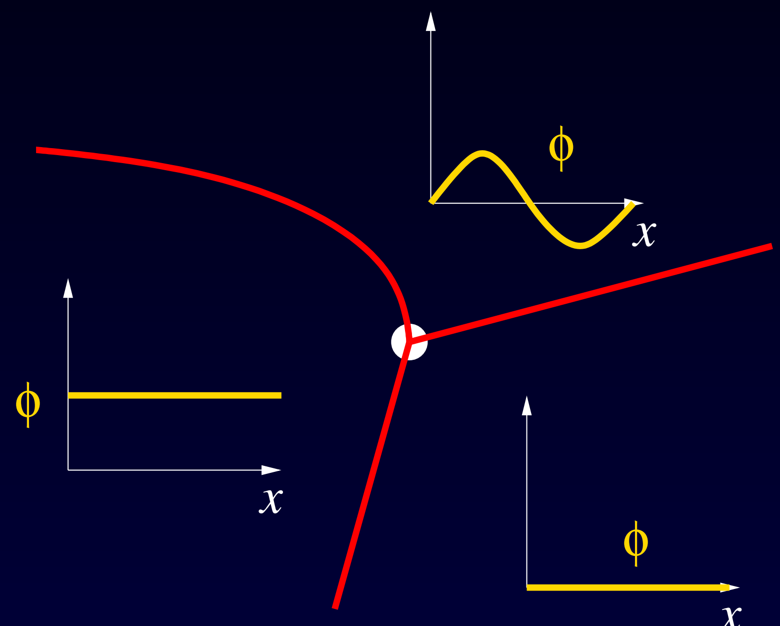
– the flow leads naturally to $z = 1$ in IR,

– accidental Lorentz invariance emerges in IR,

– the relevant coupling μ is the emergent speed of light (indeed, define $x^0 = \mu t$).

Tricriticality

The Lifshitz scalar describes a tricritical point, connecting the $\phi = 0$ and $\phi = \mathrm{const}$ phases with the spatially modulated phase.



[Lifshitz,1941]

More general Lifshitz scalars

A natural sequence of generalizations exists:

Split x^i into groups,

$$x^{i_1}, x^{i_2}, \ldots x^{i_n},$$

with i_1 $(i_2, \ldots i_n)$ taking D_1 $(D_2, \ldots D_n)$ values, and $D = D_1 + D_2 + \ldots D_n$.

Each k-th group has its own value of z, equal to k.

This leads to Lifshitz models with nested spatial anisotropy.

Gravity at a Lifshitz point

Goal: Repeat the construction for gravity.
Minimal starting point: fields $g_{ij}(t, \mathbf{x})$ (the spatial metric), action $S = S_K - S_V$, with the kinetic term

$$S_K = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g}\, \dot{g}_{ij} G^{ijk\ell} \dot{g}_{k\ell}$$

where $G^{ijk\ell} = g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell}$ is the De Witt metric, and the "potential term"

$$S_V = \frac{1}{4\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g}\, \frac{\delta W}{\delta g_{ij}} \mathcal{G}_{ijk\ell} \frac{\delta W}{\delta g_{k\ell}}$$

with W an action for g_{ij} in D Euclidean dimensions.

Gauge symmetries

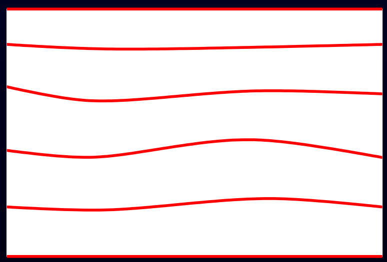
A good starting point, but: only invariant under $\tilde{x}^i = \tilde{x}^i(x^j)$, fields are only spatial metric components.

Covariantization of action:

(1) Introduce ADM-like variables N (lapse) and N_i (shift);

(2) Replace $\dot{g}_{ij} \to K_{ij} = \frac{1}{N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right)$,
Replace $\sqrt{g} \to N\sqrt{g}$.

Symmetry: Foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(M)$.
$\tilde{t} = \tilde{t}(t), \ \tilde{x}^i = \tilde{x}^i(t, x^j)$.



N and N_i are gauge fields of $\mathrm{Diff}_{\mathcal{F}}(M)$.
In the minimal realization, N is a function of only t.

First example: $z = 2$ gravity

Taking the Einstein-Hilbert action

$$W = \frac{1}{\kappa_W^2} \int d^D\mathbf{x} \sqrt{g}\, R,$$

we get

$$S = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g} N \left\{ K_{ij}K^{ij} - \lambda K^2 - \frac{\#}{\kappa_W^4} \left(R^{ij} - \frac{1}{2} R g^{ij} \right)^2 \right\}.$$

Shift in the critical dimension, as in the Lifshitz scalar:
$[\kappa^2] = 2 - D$.
The simplest model with $N(t)$ has the usual number of graviton polarizations plus an extra scalar DoF and dispersion relation $\omega^2 \sim k^4$.

Comments on detailed balance

The role of the condition of detailed balance is twofold:

(1) A technical one: Reduces the number of independent couplings in the action.
In condensed matter, nongravitational examples of theories with detailed balance exhibit a simpler renormalization structure.

(2) Perhaps a more conceptual one: The condition of detailed balance arises in systems out of equilibrium, relating S to the equilibrium theory described by W.

Detailed balance can be softly broken, or eliminated altogether, in favor of the most general action of the effective field theory approach.

Mathematical applications?

Using detailed balance, the theory is related (in imaginary time) to the covariantized Ricci flow equation,

$$\dot{g}_{ij} = \gamma N(R_{ij} + \alpha R g_{ij}) + \nabla_i N_j + \nabla_j N_i.$$

In particular, the topological version of this theory represents a natural quantum field theory associated with the Ricci flow.

Ricci flow has been instrumental in Perelman's proof of the Poincaré conjecture.

Observables and their correlation functions?

Application: Membranes at quantum criticality

The critical dimension of $z = 2$ gravity has shifted, from $1 + 1$ of the relativistic theory to $2 + 1$. This makes it a candidate theory on the worldvolume of a nonrelativistic membrane.

This theory is closely related to the bosonic string:
The groundstate wavefunction on a membrane of topology $\Sigma_h \times \mathbf{R}$ is designed to reproduce the density of the Polyakov path integral in critical string theory on worldsheet Σ_h.

The second-quantized membrane ground state which reproduces the perturbative string partition function takes the form of a Bose-Einstein condensate of the ground states of membranes of genus h, correlated across all genera.

Application: Membranes at quantum criticality

The $z = 2$ gravity theory with detailed balance simplifies in $2 + 1$ dimensions:

the potential term for pure gravity,

$$S_V \sim \int (R^{ij} - \frac{1}{2} R g^{ij})^2 \equiv 0,$$

vanishes identically. When coupled to Lifshitz scalars, S_V takes the form

$$S_V \sim \int (T^{ij})^2.$$

In order to match the Polyakov path integral of critical string theory, we need to match the symmetries, including conformal invariance.

Anisotropic Weyl symmetry

Perhaps surprisingly, a local version of anisotropic scaling symmetries can be defined, in a way compatible with $\mathrm{Diff}_{\mathcal{F}}(M)$.

For general values of z, we define

$$g_{ij} \to \exp(2\Omega(t,\mathbf{x}))g_{ij}, \quad N_i \to \exp(2\Omega(t,\mathbf{x}))N_i,$$

$$N \to \exp(z\Omega(t,\mathbf{x}))N.$$

Such anisotropic Weyl transformations represent a classical symmetry of $z=2$ gravity in $2+1$ dimensions if $\lambda = 1/2$.

More generally, anisotropic Weyl transformations:

– fix $\lambda = 1/D$;
– make $N(t,\mathbf{x})$ a function of spacetime;
– are compatible with foliation-preserving diffeomorphisms!

$z = 3$ gravity in $3+1$ dimensions

Changing z shifts the critical dimension where $[\kappa^2] = 0$.

$3+1$ dimensions require $z = 3$.

A theory with detailed balance can be written down, starting with

$$W = \int \omega_3(\Gamma(g)) \equiv \int \Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma.$$

EoM yield $C^{ij} = 0$, where

$$C^{ij} = \varepsilon^{ik\ell}\nabla_k \left(R^j_\ell - \frac{1}{4} R \delta^j_\ell \right)$$

is the ADM-Cotton-York tensor.

C^{ij} exhibits many interesting math and physics properties. For example, it is a conformal tensor.

RG flows and $z = 1$ in the IR

Theory with $z > 1$ represents a candidate UV description; barring exact conformal invariance, relevant deformations will be generated by quantum corrections.

Relevant deformations with detailed balance:

Deform W, to

$$W = \int \omega_3 + \mu \int d^D\mathbf{x}\sqrt{g}(R - 2\Lambda_W).$$

This will yield relevant terms in S,

$$S = \ldots + \int dt\, d^D\mathbf{x}\sqrt{g}N\left(\ldots + R^{ij} - \frac{1}{2}Rg^{ij} + \Lambda_W g^{ij}\right)^2$$

$$= \ldots + \int dt\, d^D\mathbf{x}\sqrt{g}N\left(\ldots R - 2\Lambda\right).$$

Or, one can break detailed balance softly, and add relevant terms directly to S.

Either way, superficially, the leading IR terms in the action match the structure of GR, with an emergent speed of light, G_N and Λ, all determined in terms of $\mu, \kappa, \ldots$.

However, there are clearly several discrepancies:

(1) $N(t)$ is only a function of time, leading to one extra propagating scalar DoF;

(2) $\lambda = 1$ in GR, but here it is a (running) coupling constant;

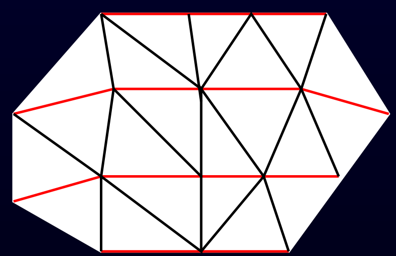
(3) gauge symmetries $\mathrm{Diff}_{\mathcal{F}}(M)$ are smaller than in (conventional) GR. While this does not necessarily represent a problem, it would be convenient to start with a theory that has the same number of local gauge redundancies as GR.

Application: Gravity on a lattice

Causal dynamical triangulations approach [Ambjørn,Jurkiewicz,Loll] to $3+1$ lattice gravity:

Naive sum over triangulations does not work (branched polymers, crumpled phases).

Modify the rules, include a preferred causal structure:



Now a continuum four-dimensional limit exists!

Spectral dimension of spacetime

More precisely, the spectral dimension is

$$d_s(IR) = 4.02 \pm 0.1$$

at long distances, and

$$d_s(UV) = 1.80 \pm 0.25$$

in the UV (before lattice artifacts are encountered).
an effective dimensional reduction to two at short distances!

The spectral dimension can be defined in the continuum framework with anisotropic scaling, yielding:

$$d_s = 1 + \frac{z}{D}.$$

$z = 4$ gravity

$z = 4$ would be formally power-counting renormalizable in $4 + 1$ dimensions. There might be, however, several reasons to consider $z = 4$ even when interested in $3 + 1$ gravity:

– the conformal factor does not get spatial dynamics in $D = 3$, $z = 3$ with detailed balance;

– in the minimal realization with the extra scalar DoF, the IR physics suggests that the scalar will be in the spatially-modulated phase.

Depending on D, the starting point is some version of

$$W = \int d^D\mathbf{x}\sqrt{g}\left(\alpha R^2 + \beta R_{ij}R^{ij} + \gamma R_{ijk\ell}R^{ijk\ell}\right).$$

Application to AdS/CFT

Anisotropic gravity systems, if consistent, could provide a new class of gravity duals for CFTs, in particular those relevant for condensed matter.

Example: Start with W which has a Euclidean AdS_D solution. Then the theory with detailed balance, described by S in $D+1$ dimensions, has a static solution given by

$$N = 1, \quad N_i = 0, \quad g_{ij} = \text{Euclidean } AdS_D \text{ metric}.$$

This geometry has an $S^{D-1} \times \mathbf{R}$ boundary.

Bulk isometries = conformal symmetries of S^{D-1} plus time translations.

These are the symmetries of a quantum critical system on the boundary, already critical in the static limit.

Ultralocal gravity

In retrospect, one example of a theory of gravity with anisotropic scaling has appeared in the literature already in the 1970's: the ultralocal theory of gravity [Isham;Teitelboim;Henneaux]

It results simply from eliminating all derivative terms from the potential, and setting

$$S_V = 2\Lambda.$$

This case can be viewed from two perspectives, either as $z = 0$ or $z = \infty$.

Remarkably, this theory is "generally covariant" – it has the same number of gauge symmetries per spacetime point as GR. The symmetry algebra is *not* that of GR, instead it is deformed into spatial diffeomorphisms and a local $U(1)$ symmetry.

Towards General Covariance

Simplest attempt: Declare N to be a function of everything, see what happens. The Hamiltonian

$$H = \int d^D\mathbf{x}\left(N\mathcal{H}_\perp + N^i\mathcal{H}_i\right).$$

This approach has worked in the ultralocal theory, leading to general covariance and the closure of the constraints.

At $1 < z < \infty$, the constraint algebra appears in trouble; it is indeed the $[\mathcal{H}_\perp(\mathbf{x}), \mathcal{H}_\perp(\mathbf{y})]$ commutator that is problematic.

One apparently consistent way of quantizing this system: With detailed balance, $\mathcal{H}_\perp$ are quadratic in "a_{ij} variables." Declaring a_{ij}'s to be the first-class constraints closes the algebra. This means quantizing the theory as a topological theory.

$U(1)$-Extended Diff$_{\mathcal{F}}$

Why do we want N to be the function of t and x^i? N is related to g_{00}, and that is where the Newton potential is.

Strategy: Keep the subleading, $\mathcal{O}(1/c^2)$ term in g_{00}:

$$g_{00} = -N(t)^2 + \frac{A_0(t, \mathbf{x})}{c^2} + \ldots,$$

and the subleading term α in the time reparametrizations as we take the $c \to \infty$ limit.

This α generates an extra $U(1)$ gauge symmetry,

$$\delta A_0 = \dot{\alpha}, \qquad \delta N_i = \partial_i \alpha, \qquad \delta g_{ij} = 0.$$

Requiring the invariance of the action under this $U(1)$-extended $\mathrm{Diff}_{\mathcal{F}}(M)$ symmetry:

(1) fixes $\lambda = 1$,

(2) requires an additional coupling

$$\int \sqrt{g} N A_0 R,$$

whose variation is

$$\propto \int \sqrt{g} N (R^{ij} - \frac{1}{2} R g^{ij}) \dot{g}_{ij} \alpha;$$

hence,

(3) the full nonlinear theory works only in $2+1$ dimensions.

This suggests a new class of anisotropic theories of gravity in $2+1$ dimensions, for arbitrary choices of S_V.

Applications of anisotropic scaling in GR and string theory

[work with Charles Melby-Thompson]

Usefulness of the concept of anisotropic scaling in gravity is not limited to "exotic" models of gravity with Lifshitz-type behavior, but extends to solutions of conventional general relativity (and string theory) with matter.

Example:

Holographic renormalization in spacetimes with "unusual" asymptotic isometries. The behavior of such geometries is often confusing near the (naively defined) boundary. In particular, Penrose's definition of conformal boundary is often limited, and often clashes with the picture expected from holography.

Anisotropic conformal infinity

Our motivation originated from the study of the global structure of Schrödinger and Lifshitz spaces.

Other useful applications: Warped AdS, near-horizon extreme Kerr geometry, etc.

Main observation: The anisotropic Weyl transformations with $z \neq 1$ are compatible with foliation-preserving diffeomorphisms.

In spacetime geometries whose asymptotic isometries are compatible with $\mathrm{Diff}_{\mathcal{F}}(M)$, one can use anisotropic conformal transformations to define an anisotropic conformal infinity/boundary of spacetime.

Global structure of Schrödinger spaces

also: [Blau et al]

Schrödinger space in Poincaré-like coordinates:

$$ds^2 = -\frac{dt^2}{u^{2z}} + \frac{2dt\, d\theta + d\mathbf{x}^2 + du^2}{u^2}.$$

(set $z = 2$ for simplicity). The Penrose conformal boundary is one-dimensional.

Define scaling: $\mathbf{x}$ has $z = 2$, θ has $z = \infty$. This means

$$t \to b^2 t, \qquad \mathbf{x} \to b\mathbf{x}, \qquad \theta \to \theta.$$

Define nested anisotropic Weyl transformations.

The anisotropic conformal boundary is as expected from holography: parametrized by $t, \mathbf{x}, \theta$ at $u = 0$.

This is confirmed by the coordinate transformations

$$\tau = \arctan t, \qquad \mathbf{y} = \frac{\mathbf{x}}{\sqrt{1+t^2}}, \qquad w = \frac{u}{\sqrt{1+t^2}},$$

$$\vartheta = \theta + \frac{t}{2(1+t^2)}(\mathbf{x}^2 + u^2),$$

to the global Schrödinger space:

$$ds^2 = -\left(1 + \frac{\mathbf{y}^2}{w^2} + \frac{1}{w^4}\right) d\tau^2 + \frac{2d\tau d\vartheta + d\mathbf{y}^2 + dw^2}{w^2}.$$

This transformation is a nested foliation-preserving diffeo!

The same definition of anisotropic conformal boundary works for the Lifshitz spaces, warped AdS_3, near-horizon extremal Kerr geometry, . . .

One can also define anisotropic geodesic motion.

Quantum Gravity with Anisotropic Scaling near the Schwarzschild Horizon

Near-horizon limits of black holes and black branes have been a very productive tool, first in SUSY extremal cases (AdS/CFT), then in extremal non-SUSY cases (Kerr/CFT); Schwarzschild?

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2G_N M}{r}\right)} dr^2 + r^2 d\Omega_2^2.$$

expand in smallness of $\rho = r - r_H$; naively,

$$ds^2 \approx -\frac{\rho}{r_H} dt^2 + \frac{r_H}{\rho} d\rho^2 + r_H^2 d\Omega_2^2.$$

This is Rindler$_2 \times S^2$, and (of course) violates Einstein's equations!

The Anisotropic Scaling

Usually, the near-horizon limit is taken as a scaling limit of the *solution*, with $M \to \infty$. We instead keep M fixed, and take the scaling limit of the *theory!*

Split x^μ into $(t,x) \equiv x^\alpha$, $\alpha = 0,1$, and $(\theta,\rho) = y^a$, with $a = 1,2$. Introduce $c_\perp$, the speed of light along the horizon, and take the $c_\perp \to 0$ limit. This leads to anisotropic scaling

$$x^\alpha \to b x^\alpha,$$

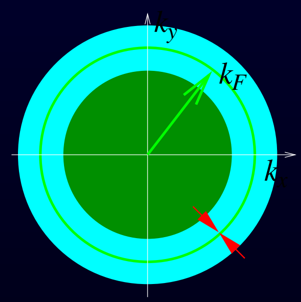
$$y^a \to y^a.$$

This is a new type of Lifshitz scaling in gravity, with a $2+(d-2)$ split of spacetime, and a spatial anisotropy. Now $z=1$ along x, and $z=\infty$ along y^a.

Recall RG picture of Landau's theory of Fermi liquids

$$S_F = \int dt\, d^D\mathbf{x} \left(i\Psi^\dagger \dot{\Psi} + \Psi^\dagger \Delta \Psi + \mu \Psi^\dagger \Psi + \ldots \right).$$

Wilsonian renormalization: Identify the ground state! Scaling towards the Fermi surface:



write $k = k_F + K$, define scaling $K \to K/b$, $k_F \to k_F$; dimensions θ along the Fermi surface $\theta \to \theta$. System effectively behaves as a collection of $1+1$ relativistic fermions, parametrized by "internal index" θ.

Anisotropic gravity near the horizon

First guess:

$$S = \int d^4x \sqrt{-g} \left({}^{(4)}R - {}^{(2)}R \right).$$

This indeed has $\mathbf{R}^2 \times \Sigma_2$ as a solution.

This theory shares many good features of the ultralocal theory:

– the action does not contain higher derivatives;
– algebra of Hamiltonian constraints simplifies (perhaps as many gauge symmetries as GR?);
– N (and, indeed, all of $g_{\alpha\beta}$) is a spacetime-dependent field.

In addition, the model is power-counting renormalizable, as a result of the spatial anisotropy. Quantum corrections are calculable; model can be extended to higher dimensions.

Future directions

Can gravity with anisotropic scaling be engineered from string theory?

Is there a version of the theory with “general covariance” and the same number of DoF as GR, and if so, can it flow to GR in the infrared?

What is the ground state of the mysterious “spatially modulated phase” of gravity?

Is the condition of detailed balance in gravity selfcontained under renormalization? What is the pattern of RG flows in this class of models?

What is the role of detailed balance condition: Is it just a technical tool, or is it related (as in condensed matter) to gravity out of equilibrium?