Symmetries and Constraints

Axel Kleinschmidt

Work with: Thibault Damour, Hermann Nicolai

References: [0709.2691] and work in progress
Central idea of $E_{10}$ approach
Central idea of $\mathbb{E}_{10}$ approach

Evidence for a correspondence [Damour, Henneaux, Nicolai 2002]

$D = 11$ supergravity solutions

Constrained null geodesics on $\mathbb{E}_{10}/K(\mathbb{E}_{10})$ coset space
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Geometry

Algebra
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Geometry

Algebra

Time plays a special role:

- Canonical framework
- Split into dynamical equations and constraint equations
Origin: cosmological billiards

Gravity near space-like singularity [Belinskii, Khalatnikov, Lifshitz 1970]

\[ \mathcal{L} = \sum_{a,b=1}^{10} n^{-1} G_{ab} \partial_t \beta^a \partial_t \beta^b + V_{\text{eff}}(\beta) \]

spatial scale factors \( \beta^a \sim -\log g_{aa} \)

Lorentzian DeWitt metric
Gravity near space-like singularity [Belinskii, Khalatnikov, Lifshitz 1970]

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  consists of infinite potential walls, obstructing free null motion of \( \beta \) variables

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- Resulting billiard geometry that of \( E_{10} \) Weyl chamber for any maximal supergravity \cite{Damour, Henneaux 2000}
Origin: cosmological billiards

Gravity near space-like singularity \[\text{[Belinskii, Khalatnikov, Lifshitz 1970]}\]

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- Close to the singularity \(V_{\text{eff}}(\beta)\)
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- Resulting billiard geometry that of \(E_{10}\) Weyl chamber for any maximal supergravity \[\text{[Damour, Henneaux 2000]}\]

\[\Rightarrow E_{10} \text{ conjecture from inclusion of more variables} \text{ [DHN]}\]
On $E_{10}$: Definition
On $\text{E}_{10}$: Definition

$\text{E}_{10}$: very special hyperbolic rank 10 Kac–Moody algebra
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$E_{10}$: very special hyperbolic rank 10 Kac–Moody algebra

- Generated from 10 basic $SL(2, \mathbb{R})$ triples $e_i, h_i, f_i$

Relations encoded in Dynkin diagram
On $E_{10}$: Definition

$E_{10}$: very special hyperbolic rank 10 Kac–Moody algebra

- Generated from 10 basic $SL(2, \mathbb{R})$ triples $e_i, h_i, f_i$
- Picture as $(\infty \times \infty)$ matrices (no simple condition!)

Positive step operators (strictly upper triangular) $e_i$
Cartan operators (diagonal) $h_i$
Negative step operators (strictly lower triangular) $f_i$

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\[
\begin{pmatrix}
\ddots & 0 & 1 \\
0 & \ddots & \ddots \\
\end{pmatrix}, \quad
\begin{pmatrix}
\ddots & 1 \\
-1 & \ddots \\
\end{pmatrix}, \quad
\begin{pmatrix}
\ddots & 0 \\
-1 & \ddots \\
0 & \ddots \\
\end{pmatrix}
\]

Positive step operators (strictly upper triangular) $e_i$
Cartan operators (diagonal) $h_i$
Negative step operators (strictly lower triangular) $f_i$

- Only known presentation of $E_{10}$

Relations encoded in Dynkin diagram
Choose $\text{GL}(10, \mathbb{R})$ subalgebra and write everything as graded list of $\text{GL}(10, \mathbb{R})$ tensors

$\text{SL}(10, \mathbb{R}) \subset \text{GL}(10, \mathbb{R}) \subset E_{10}$
Choose $\text{GL}(10, \mathbb{R})$ subalgebra and write everything as graded list of $\text{GL}(10, \mathbb{R})$ tensors

Result [DHN]

$$E_{10} = \langle \ldots, F_{a_1 a_2 a_3}, K^{a}_b, E^{a_1 a_2 a_3}, E^{a_1 \ldots a_6}, E^{a_0 | a_1 \ldots a_8}, \ldots \rangle$$
On $E_{10}$: Level decomposition

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\[ \ell = -1 \quad \ell = 0 \quad \ell = 1 \quad \ell = 2 \quad \ell = 3 \]

$\text{SL}(10, \mathbb{R}) \subset \text{GL}(10, \mathbb{R}) \subset E_{10}$
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Choose $GL(10, \mathbb{R})$ subalgebra and write everything as graded list of $GL(10, \mathbb{R})$ tensors

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$SL(10, \mathbb{R}) \subset GL(10, \mathbb{R}) \subset E_{10}$
The $E_{10}/K(E_{10})$ coset model
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Compact subalgebra: $K(E_{10}) = \{ x \in E_{10} \mid x^T = -x \}$

Think of
- $E_{10}$ infinite generalisation of $GL(10, \mathbb{R})$
- $K(E_{10})$ infinite generalisation of $SO(10)$
The $\text{E}_{10}/\text{K}(\text{E}_{10})$ coset model

Compact subalgebra: $\text{K}(\text{E}_{10}) = \{x \in \text{E}_{10} \mid x^T = -x\}$

Think of

$\bullet \text{E}_{10}$ infinite generalisation of $\text{GL}(10, \mathbb{R})$

$\bullet \text{K}(\text{E}_{10})$ infinite generalisation of $\text{SO}(10)$

$\text{E}_{10}/\text{K}(\text{E}_{10})$ coset element $\mathcal{V}$ in triangular gauge

$$\mathcal{V}(t) = e_m^a(t) \exp \left[ \sum_{\ell > 0} A(\ell)(t) \ast E(\ell) \right]$$

$t \in \mathbb{R}$
The $\text{E}_{10}/K(\text{E}_{10})$ coset model

Compact subalgebra:  

$$K(\text{E}_{10}) = \{ x \in \text{E}_{10} \mid x^T = -x \}$$

Think of

- $\text{E}_{10}$ infinite generalisation of $\text{GL}(10, \mathbb{R})$
- $K(\text{E}_{10})$ infinite generalisation of $\text{SO}(10)$

$\text{E}_{10}/K(\text{E}_{10})$ coset element $\mathcal{V}$ in triangular gauge

$$\mathcal{V}(t) = e_m^a(t) \exp \left[ \sum_{\ell > 0} A(\ell)(t) \star E^{(\ell)} \right] \quad (t \in \mathbb{R})$$

Spatial tetrad

$$e_m^a \in \text{GL}(10, \mathbb{R})/\text{SO}(10)$$

One field per positive level operator

Worldline parameter
Null geodesic equation on $E_{10}/K(E_{10})$

\[ \mathcal{D}\mathcal{P} := \partial_t \mathcal{P} + [Q, \mathcal{P}] = 0 \quad , \quad \langle \mathcal{P} | \mathcal{P} \rangle = 0 \]

where \( \nu^{-1} \partial_t \nu = Q + \mathcal{P} \)

velocity along coset

\( K(E_{10}) \) valued connection
Null geodesic equation on $E_{10}/K(E_{10})$

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velocity along coset $K(E_{10})$ valued connection

Invariances

- Global $E_{10}$
- Local $K(E_{10})$

$$\mathcal{V} \rightarrow g \mathcal{V} k(t)^{-1}$$
Null geodesic equation on $E_{10}/K(E_{10})$

\[ \mathcal{D}\mathcal{P} := \partial_t \mathcal{P} + [Q, \mathcal{P}] = 0 \quad , \quad \langle \mathcal{P} | \mathcal{P} \rangle = 0 \]

where \( \mathcal{V}^{-1} \partial_t \mathcal{V} = Q + \mathcal{P} \) \( \xrightarrow{\text{velocity along coset}} \) \( K(E_{10}) \) valued connection

- Invariances
  - Global $E_{10}$
  - Local $K(E_{10})$

Equation is (Toda) integrable! Conserved charge: $\partial_t \mathcal{I} = 0$

**Strategy:** Evaluate equation in level decomposition. Can be truncated consistently after a given level.
Central result [DHN]

Correspondence between dynamical SUGRA equations in truncation and geodesic equations up to 'height' 29 (beyond billiard approximation). Requires gauge-fixing.
Central result \[\text{[DHN]}\]

Correspondence between \textit{dynamical} SUGRA equations in truncation and geodesic equations up to ‘height’ 29 (beyond billiard approximation). Requires gauge-fixing

\textit{‘Dynamical’}: Equations containing second time derivatives

\[ \varepsilon_{ab} := R_{ab} - \frac{1}{2} \delta_{ab} R - T_{ab} = 0 \]

\[ \mathcal{M}^{a_1 a_2 a_3} := D_K F^{K a_1 a_2 a_3} - CS = 0 \]

\[ D[0 F_{a_1 a_2 a_3 a_4}] = 0 \]

\[ R[0 a b c] = 0 \]
The $E_{10}$ correspondence

Central result [DHN]

Correspondence between dynamical SUGRA equations in truncation and geodesic equations up to 'height' 29 (beyond billiard approximation). Requires gauge-fixing

'Dynamical': Equations containing second time derivatives

\[
\mathcal{E}_{ab} := R_{ab} - \frac{1}{2} \delta_{ab} R - T_{ab} = 0
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\[
\mathcal{M}^{a_1 a_2 a_3} := D_K F^{K a_1 a_2 a_3} - CS = 0
\]

Incomplete!
Misses constraints

\[
\mathcal{E}_{0a} = 0 \quad \text{(Diff)}
\]

\[
\mathcal{M}^{0a_1 a_2} = 0 \quad \text{(Gauss)}
\]

\[
D[0 F_{a_1 a_2 a_3 a_4}] = 0
\]

\[
R[0a b]_c = 0
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\[
\ldots
\]
Constraints: Generalities

Two possible approaches to constraints
Constraints: Generalities

Two possible approaches to constraints

1. Study SUGRA constraints, use $E_{10}$ correspondence to obtain coset constraints

Prove consistency with geodesic equation
Constraints: Generalities

Two possible approaches to constraints

1. Study SUGRA constraints, use $E_{10}$ correspondence to obtain coset constraints
   - Prove consistency with geodesic equation

2. Try to define constraints directly in coset model
   - Uncover algebraic structure
   - Verify consistency with SUGRA
Constraints: Generalities

Two possible approaches to constraints

1. Study SUGRA constraints, use $E_{10}$ correspondence to obtain coset constraints
   - Prove consistency with geodesic equation

2. Try to define constraints directly in coset model
   - Uncover algebraic structure
   - Verify consistency with SUGRA

Both approaches followed in [Damour, AK, Nicolai 2007].
Here focus on 2.
Constraints: Ansatz
Constraints: Ansatz

Know: conserved charges $\mathcal{J}$ from global $E_{10}$ symmetry. For correspondence had to truncate both systems. Implies

$$\mathcal{J} = (\ldots, 0, (\frac{3}{2})^{n_0|n_1\ldots n_8}, (\frac{2}{2})^{n_1\ldots n_6}, (\frac{1}{2})^{n_1 n_2 n_3}, (0)^{n_m}, \ldots)$$

Everything zero  Everything non-zero
Constraints: Ansatz

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$$\mathcal{J} = (\ldots, 0, (\mathcal{J} n_0|n_1\ldots n_8, (\mathcal{J} n_1\ldots n_6, (\mathcal{J} n_1 n_2 n_3, (0) n m, \ldots))$$

Everything zero  Everything non-zero

$\Rightarrow$ Ansatz: Constraints quadratic in $\mathcal{J}$, structure as SUGRA

$$\mathcal{L}_{(-3)} n_1\ldots n_9 = (\mathcal{J} p|[n_1\ldots n_8 (0) n_9] p + \alpha (\mathcal{J} n_1 n_2 n_3 (\mathcal{J} n_4\ldots n_9$$

$$\mathcal{L}_{(-4)} [n_1\ldots n_{10}]|m_1 m_2 = (\mathcal{J} n_1 n_2 [m_1 (\mathcal{J} m_2)|n_3\ldots n_{10}$$

$$-\beta (\mathcal{J} n_1\ldots n_5[m_1 (\mathcal{J} m_2]n_6\ldots n_{10}$$

$\ldots$


Constraints: General structure

\[ \mathcal{L} = \sum_{m \geq 0} (-\ell + m)_J \ast (-m)_J \]
Constraints: General structure

\[ \mathcal{L} = \sum_{m \geq 0} (-\ell + m) J * (-m) J \]

Remark: Truncation (to \( \ell \leq 3 \)) implies finite sums. Also breaking of \( E_{10} \) to Borel subgroup \( E_{10}^+ \).
Constraints: General structure

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- Free coefficients? Fixed by requiring simple transformation under \( E_{10}^+ \).
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- Tensor structure? Fixed by comparison to supergravity and/or by \( E_{10} \) representation theory.
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- **Free coefficients?** Fixed by requiring simple transformation under \( E_{10}^+ \).
- **Tensor structure?** Fixed by comparison to supergravity and/or by \( E_{10} \) representation theory.
- **Consistency?** Applying correspondence gives correct supergravity constraints.
Constraints: Result

Outcome [Damour, AK, Nicolai]

Set of four constraints $\mathcal{L}^{(\ell)}$ (for $-6 \leq \ell \leq -3$), preserved by geodesic motion. Correspond to SUGRA constraints.
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$(\ell)$

- $\mathcal{L}$ Spatial diffeomorphism $\mathcal{E}_{0a}$
- $\mathcal{L}$ Gauss constraint $\mathcal{M}_{0a_1a_2}$
- $\mathcal{L}$ Bianchi identity $D[a_1 F_{a_2...a_5}]$
- $\mathcal{L}$ Riemann constraint $R[a_1 a_2 a_3] a_0$
Constraints: Result

Outcome [Damour, AK, Nicolai]

Set of four constraints $\mathcal{L}^{(\ell)}$ (for $-6 \leq \ell \leq -3$), preserved by geodesic motion. Correspond to SUGRA constraints.

- $(-3)\mathcal{L}$: Spatial diffeomorphism $\mathcal{E}_{0a}$
- $(-4)\mathcal{L}$: Gauss constraint $\mathcal{M}_{0a_1a_2}$
- $(-5)\mathcal{L}$: Bianchi identity $D[a_1 F_{a_2 \ldots a_5}]$
- $(-6)\mathcal{L}$: Riemann constraint $R[a_1 a_2 a_3] a_0$

In addition: Hamiltonian constraint $\mathcal{E}_{00} \leftrightarrow \langle \mathcal{J} | \mathcal{J} \rangle$
**Constraints: Remarks**

- **$E_{10}$ representation theory**: Tensor structure and transformation related to integrable highest weight representation of $E_{10}$, highest weight

  \[ \Lambda_1 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \]

  First step in an extension of $E_{10}$ to $E_{11}$ [AK 2003].
  Relation to $E_{11}$ programme of [West]? Also [Englert, Houart]
### Constraints: Remarks

- **$E_{10}$ representation theory**: Tensor structure and transformation related to integrable highest weight representation of $E_{10}$, highest weight

  \[ \Lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

  First step in an extension of $E_{10}$ to $E_{11}$ [AK 2003].

  Relation to $E_{11}$ programme of [West]? Also [Englert, Houart]

- **Space-time**: Coset model has only $t$-dependent $K(E_{10})$ symmetry, containing local spatial $SO(10)$ Lorentz transformations and $t$-dep. gauge transformations. Global $E_{10}$ can be thought of as linear in $x$.

  Constraints generate additional transformations: Spatial diffeomorphisms, space-dependent gauge transformations etc. \[ \Rightarrow \text{Emergent space} \]
<table>
<thead>
<tr>
<th>An analogy</th>
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Axel Kleinschmidt
An analogy

String theory

- Conf. gauge equations

\[ \partial^2 X^\mu = 0 \]

Free \(\Rightarrow\) simple!

\(\text{E}_{10}\) coset model
An analogy

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Axel Kleinschmidt
Analogy pursued
Analogy pursued

$E_{10}$ has affine $E_9$ subalgebra. Restricting the constructed $\mathcal{L}^{(\ell)}$ to $E_9$ yields exactly one non-zero constraint: the Virasoro-Sugawara $L_{-1}$. Generates translations of the affine spectral parameter.
Analogy pursued

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Questions raised by this analogy

- What is the analogue of conformal symmetry? Of the Polyakov action?
- Spectral parameter for hyperbolic algebras? [Evidence also from fermions [AK, Nicolai, Palmkvist 2006].]
- Is there a Sugawara construction for $E_{10}$?
- Constraint algebra $\{\mathcal{L}, \mathcal{L}\}_{\text{P.B.}} \sim \sum J \mathcal{L}$?
Analogy pursued

$E_{10}$ has affine $E_9$ subalgebra. Restricting the constructed $\mathcal{L}^{(\ell)}$ to $E_9$ yields exactly one non-zero constraint: the Virasoro-Sugawara $L_{-1}$. Generates translations of the affine spectral parameter.

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Answers not known...
Summary and generalizations
Summary and generalizations

Presented an algebraic approach to supergravity

- Symmetry based ($\mathbb{E}_{10}$). Canonical
- Spaceless model. Integrable
- Space and diffeomorphisms emergent via constraints
Summary and generalizations

Presented an algebraic approach to supergravity

- Symmetry based ($E_{10}$). Canonical
- Spaceless model. Integrable
- Space and diffeomorphisms emergent via constraints

Generalizations and other aspects studied

- Embedding of IIA and IIB supergravity [AK, Nicolai 2004]
- Deformations (mass [Henneaux, Jamsin, AK, Persson 2008], gauging [Bergshoeff et al. 2007 & 2008][Riccioni, West 2007])
- Fermions [Damour, AK, Nicolai 2005][de Buyl, Henneaux, Paulot 2005]
- Curvature corrections [Damour, Nicolai 2005]
- Exact solutions (S-branes) [AK, Nicolai 2005]
- Theories with less supersymmetry
Open problems

- Completion of correspondence
  ⇒ Interpretation of higher levels?

- Sugawara construction and spectral parameter
  ⇒ Emergence of space?

- Supersymmetry on the worldline
  ⇒ Green–Schwarz mechanism for space-time?

- Quantisation and arithmetic properties
  ⇒ Weyl group $\mathcal{W}(E_{10}) = \text{PGL}_2(O)$? [Feingold, AK, Nicolai 2008]

- Duality symmetries?

- ...
(Speculative) Outlook
(Speculative) Outlook

Geometry \leftrightarrow E_{10} \text{ correspondence} \leftrightarrow \text{Algebra}
(Speculative) Outlook

Geometry \leftrightarrow \text{ Algebra}

\text{Quantisation}

\text{Number theory}

\mathcal{E}_{10} \text{ correspondence}
(Speculative) Outlook

Geometry \quad \longleftrightarrow \quad Algebra

\begin{align*}
\text{E}_{10} \text{ correspondence} \\
??? \quad \longleftrightarrow \quad \text{Number theory} \\
\text{Quantisation}
\end{align*}
(Speculative) Outlook

Geometry \[\xrightarrow{E_{10}\text{ correspondence}}\] Algebra

??? \[\xleftarrow{\text{Quantisation}}\] Number theory

Thank you for your attention!
Extra Slides
Inclusion of fermions requires $K(E_{10})$ representation theory

- $K(E_{10})$ contains $SO(10)$ Lorentz subgroup
- But: $K(E_{10}) \in$ unknown class of groups, not Kac–Moody
- $\exists$ genuine $K(E_{10})$ representations! Most useful:
  - $320$ vector-spinor $\Psi$ and $32$ Dirac-spinor $\epsilon$
- Finite-dimensional hence unfaithful: $K(E_{10})$ not simple
- Results in spinning particle on $E_{10}/K(E_{10})$

$$\mathcal{L} = \frac{1}{2n} \langle \mathcal{P} | \mathcal{P} \rangle - i \langle \Psi | D \Psi \rangle_{vs}$$

Not fully supersymmetric!

- Analogue of supersymmetry constraint: $\mathcal{S} := \mathcal{P} \circ \Psi = 0$
**Dictionary**

**GL(10, \mathbb{R})** level decomposition of $E_{10}$ implies

\[
P = (P_{(ab)}, P_{a_1a_2a_3}, P_{a_1...a_6}, P_{a_0|a_1...a_8}, \ldots)
\]

\[
P_{ab} = -N\omega_{a0b0}
\]

\[
P_{a_1...a_6} = -\frac{N}{4!}\epsilon_{a_1...a_6b_1...b_4}F_{b_1...b_4}
\]

\[
P_{a_1a_2a_3} = NF_{0a_1a_2a_3}
\]

\[
P_{a_0|a_1...a_8} = \frac{3N}{4}\epsilon_{a_1...a_8b_1b_2}\Omega_{b_1b_2a_0}
\]

- **Gauge fixed** $D = 11$ vielbein
- **Temporal gauge:** $A_{tmn} = 0$

\[
E^A_M = \begin{pmatrix}
N & 0 \\
0 & e^a_m
\end{pmatrix}
\]

- **Traceless spin connection**
  \[
  \omega_{a_{ab}} = 0
  \]
Versatility: Different subgroups

Many subgroups of $E_{10}$ that can be chosen

- mIIA supergravity
  
  \[
  \text{GL}(9, \mathbb{R})_A
  \]

- IIB supergravity
  
  \[
  \text{GL}(9, \mathbb{R})_B
  \]

- $\mathcal{N} = 8$ in $D = 4$
  
  \[
  \text{GL}(3, \mathbb{R})
  \]

- Apparently also encoded: Deformations (gaugings)
  
  [Riccioni, West 2007][Bergshoeff et al. 2007]

- mIIA supergravity
  
 Apparently also encoded: Deformations (gaugings)

- IIB supergravity
  
  Apparently also encoded: Deformations (gaugings)

- $\mathcal{N} = 8$ in $D = 4$
  
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[Note: Diagram showing relationships between groups and subgroups with arrows and labels.]

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ULB

Axel Kleinschmidt
Higher level fields

Level decomposition up to \( \ell = 28 \) [Fischbacher, Nicolai 2003]

Possible interpretations of \( \sim 4 \times 10^9 \) representations?

- String modes? Unlikely...
- Supergravity modes? Two classes (cf. [Riccioni, West 2006])

Fields dual to SUGRA d.o.f.s

\[ \text{cf. Geroch group} \]

Deformation parameters

\[ \text{Gaugings, mass} \]

Gradient conjecture [DHN]

Tensor hierarchies

[de Wit, Samtleben 2005]
[de Wit, Nicolai, Samtleben 2008]

More likely...