Null polygonal Wilson loops and scattering amplitudes via minimal surfaces in AdS

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Based on work by F. Alday and J. M.

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0903.4707 , 0904.0663

Strings 2009, Rome

N=4 SYM

- N=4 SYM is the ``simplest'' gauge theory
- We can do higher loop computations in N=4 than in any other gauge theory
- We have many results at weak and strong coupling
- The planar theory is integrable. Some exact results.

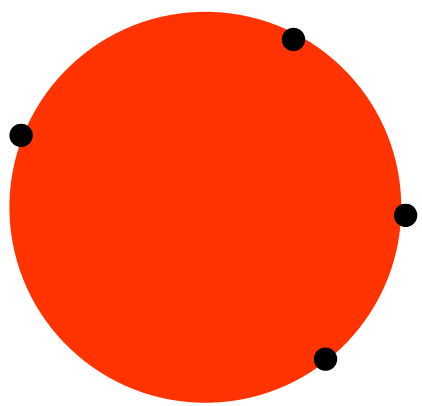
Integrability

- We can compute certain quantities exactly.
- Need to develop tricks to do it.

- We hope that these exact solutions will shed some light on how the gauge/string duality works.
  - Spectrum
  - Correlation functions

Amplitudes

- 4 dimensional scattering amplitudes are an interesting (quasi) observable of the four dimensional theory.
- Disk diagram in the string theory



Amplitudes at strong coupling

- Strong coupling has a description in terms of a simple string theory in $AdS_5 \times S^5$

- The knowledge of both weak and strong coupling helps in finding them for all coupling

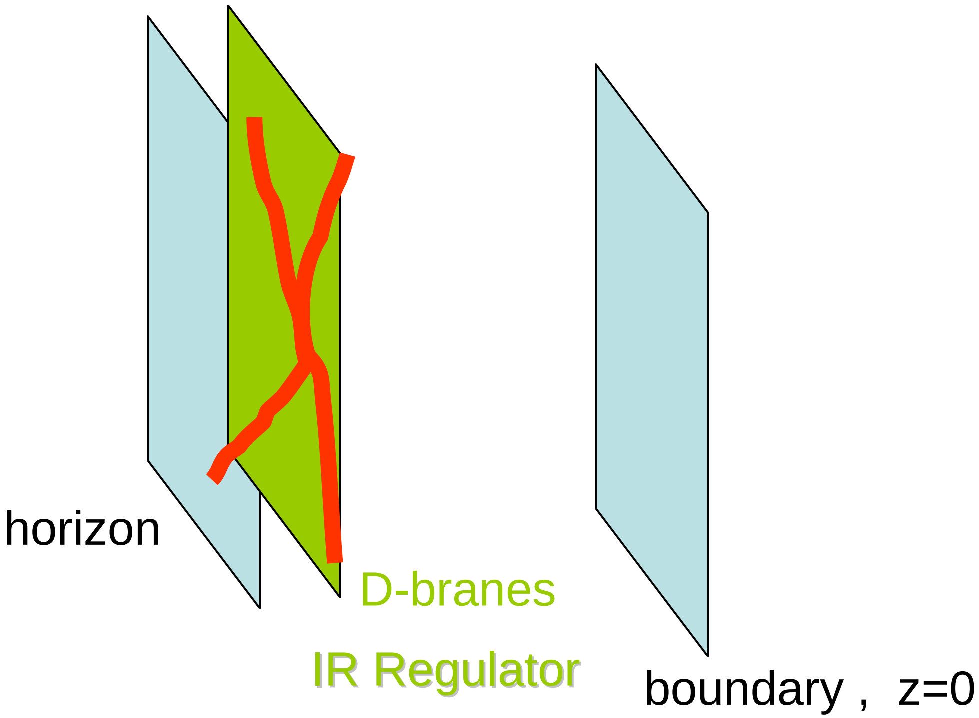
- The extra symmetries associated to integrability are easier to see at strong coupling

Note: - N=4 SYM is really different than QCD at strong coupling.
- Large IR divergencies at strong coupling → hard to set up the experiment to see them (very suppressed).

Some symmetries which are simple to see at strong coupling are also present at weak coupling.

Amplitudes at Strong coupling

Alday & JM



Original

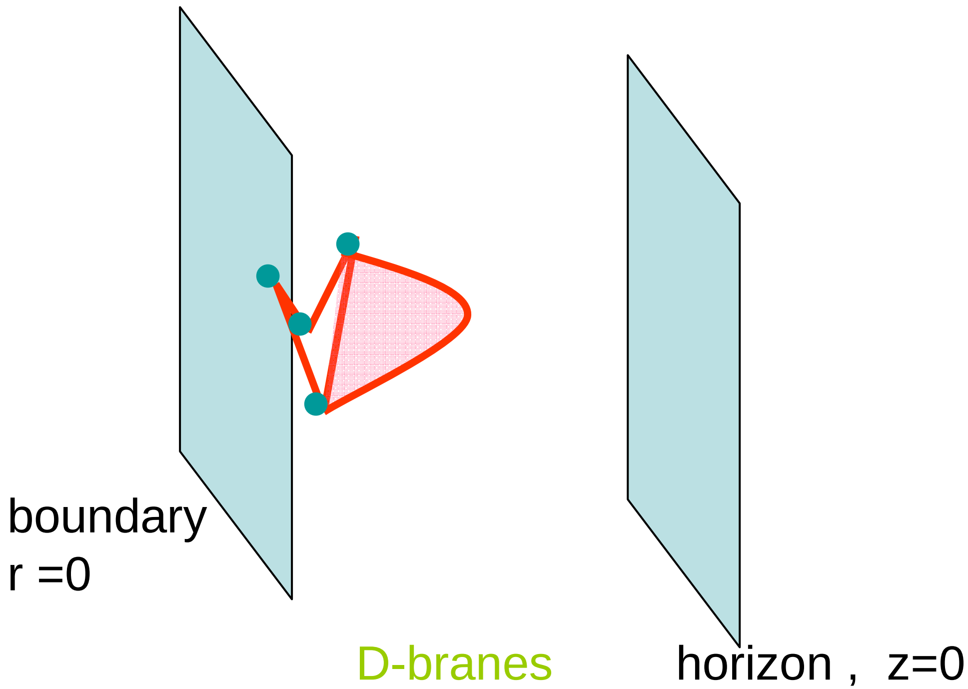
$$ds^2 = \frac{dx^2 + dz^2}{z^2}$$

T – duality:

$$d\,y = *_2 \frac{dx}{z^2}\ , \qquad r = \frac{1}{z}$$

T-dual

$$ds^2 = \frac{dy^2 + dr^2}{r^2}$$

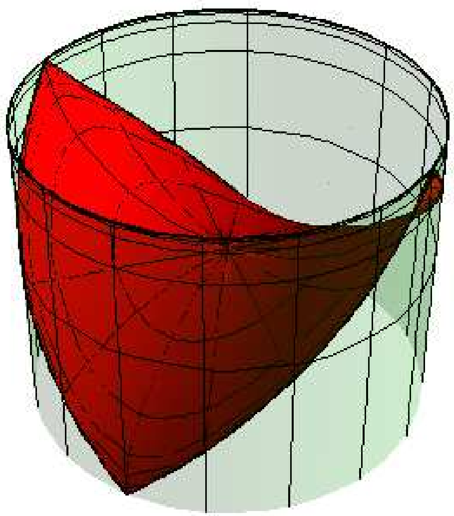


Wilson loops

Area of a minimal surface in AdS that ends on the polygon

$$\mathcal{A} \sim \langle W \rangle \sim e^{-\frac{R^2}{2\pi\alpha'}(\mathrm{Area})} = e^{-\frac{\sqrt{\lambda}}{2\pi}(\mathrm{Area})}$$

Depends on
the kinematics



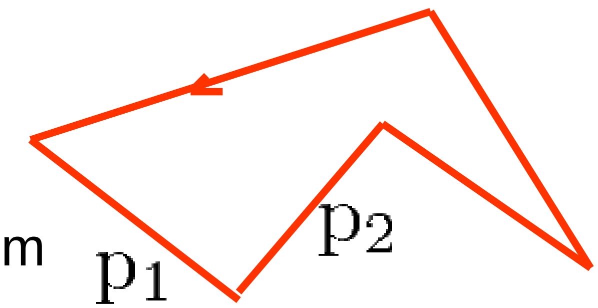
Original AdS ⟷ Dual $\widetilde{\mathrm{AdS}}$

SO(2,4) ⟷ Dual $\widetilde{\mathrm{SO}(2,4)}$

New Symmetry → More constraints on the amplitude

Amplitudes = Wilson loop, null polygon

- Polygon
- null sides
- each side= momentum



The new symmety is related to integrability

Beisert, Ricci, Tseytlin
Wolf
N. Berkovits & JM

Bosonic + fermionic T-duality → We have dual conformal symmetry to all orders in α' perturbation theory.

Dual symmetries = higher charges of the sigma model + the ordinary ones → Infinite number of conserved charges.

We have argued that the two conformal symmetries are present in the quantum theory → integrability in the quantum theory.

Constraints of dual conformal symmetry

Bern Dixon Smirnov

One loop result

(up to single log divergencies)

$$\frac{A_{MHV}}{A_{MHV,Tree}} \stackrel{?}{=} \langle W \rangle = \exp\left[\Gamma_{cusp}(\mathrm{Div} + \mathrm{BDS}) + R\right]$$

IR divergent piece

cusp anomalous dimension

Finite remainder Function of cross ratios

Drummond, Henn, Korchemsky, Sokatchev

Ward identities for broken dual conformal symmetry can be proven in the Wilson loop side

Cross ratios

Define y coordinates: $$p_i^\mu = y_i^\mu - y_{i+1}^\mu$$

Cross ratios $$\chi = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}$$ Any four points

N=4, 5 no cross ratios → 4 and 5 point amplitudes computed for all values of the coupling.

Large N is harder…

N=6 3 cross ratios → 2 loops: Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich / Drummond, Henn, Korchemsky, Sokatchev. Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini

N=7 6 cross ratios

N=8 9 cross ratios ← Analyze this at strong coupling

(N = number of gluons)

Exploring the remainder function R

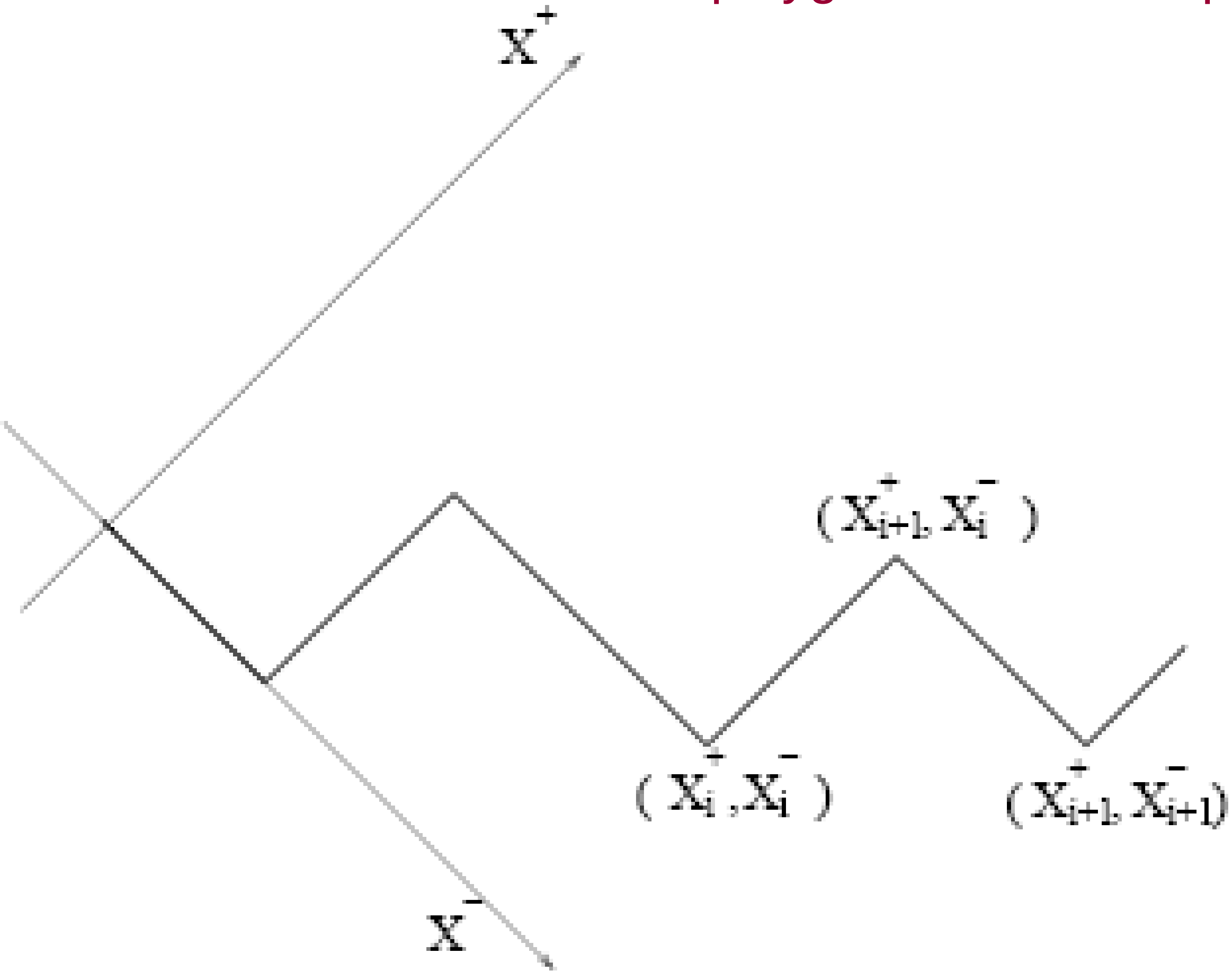
- Look at the dependence on only some of the variables.

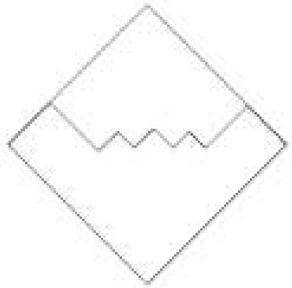
Special kinematics

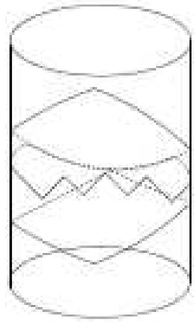
- Particles with momentum in 1 +1 dimensions. (Full 3+1 dimensional theory in the loops.)
- At strong coupling → the string surface lives in AdS_3

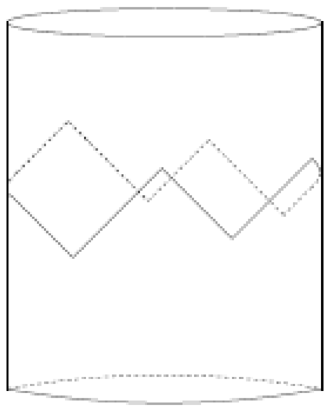
$$p_i^{\pm} \qquad i = 1, \cdots n \qquad\qquad N = 2\, n$$

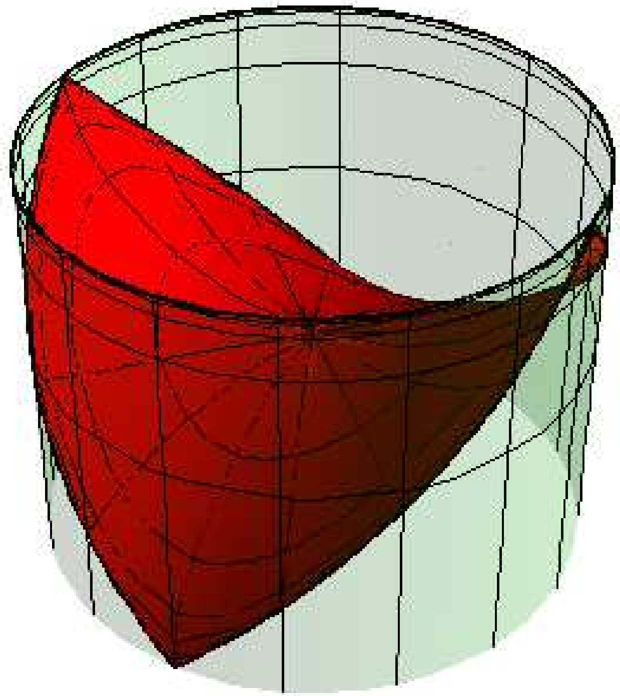
Null polygonal Wilson loop in $R^{1,1}$











Wilson loops in $R^{1,1}$

- Number of gluons $N = 2n$
- n-3 plus cross ratios x_i^+
- n-3 minus cross ratios x_i^-

Conformal group SO(2,2) = SL(2) x SL(2) = 3 + 3 generators

6 gluons, 6 sides , no cross ratio

8 gluons, 8 sides (n=4) , one cross ratio of each kind, 2 cross ratios.

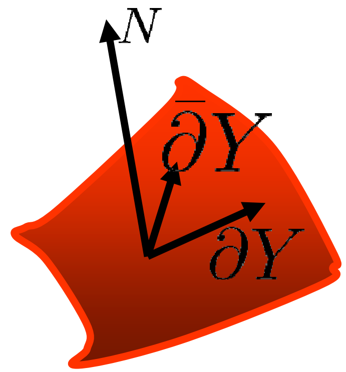
In 4 dimensions N=8 had 9 cross ratios , here we are exploring a 2 dimensional subspace

Strings in AdS$_3$

Pohlmeyer,
Jevicki, Jin, Kalousios, Volovich

$$Y^M \;, \quad -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$$

$$Y^M, \quad \partial Y^M, \quad \bar\partial Y^M \;, \quad N^M = \text{normal}$$



$$e^\alpha = \partial Y . \bar\partial Y \;, \quad p(z) = \partial^2 Y . N \;, \quad \bar p(\bar z) = \bar\partial^2 Y . N$$

equations of motion + Virasoro constraints

$$\bar\partial \partial \alpha - e^\alpha + e^{-\alpha} |p|^2 = 0$$

Generalized Sinh Gordon equation

$$\text{Area} = \int d^2 z e^\alpha$$

p : Holomorphic function
α single degree of freedom

Explicitly SO(2,2) invariant in target space

Recovering the spacetime coordinates

Linear problem

$$\mathrm{d}\,\psi^L + B^L \psi^L = 0 \qquad d\psi^R + B^R \psi^R = 0$$

$\mathrm{B}^L, \quad B^R$ are 2 x 2 matrices that depend on α and p

e.g. $\begin{pmatrix} \partial\alpha & e^{\alpha} \\ pe^{-\alpha} & -\partial\alpha \end{pmatrix}$

we have two solutions for each equation.

$$\psi^L_a \ , \qquad \psi^R_{\dot{a}}$$

$$\mathrm{Y}_{a\dot{a}} = (\psi^L_a)^t \psi^R_{\dot{a}}$$

We recover the spacetime coordinates from the solutions

$a, \ \dot{a}$ = 1, 2 are spacetime indices

Compare to $g = g_L(z) g_R(\bar{z})$ for WZW models

- What boundary conditions ?

- Worldsheet: whole complex plane
- p = polynomial

$$p = z^{n-2} + m_{n-4} z^{n-4} + \cdots m_0$$

(For 2 n gluons)

Number of non-trivial coefficients of p is n-3 → 2 (n-3) real parameters

Recipe

1. Choose p — $p = z^{n-2} + \cdots$
2. Solve Sinh-Gordon equation — $\bar{\partial}\partial\alpha - e^{\alpha} + e^{-\alpha}|p|^2 = 0$
3. Solve the linear problems — $d\psi^L + B^L\psi^L = 0 \qquad d\psi^R + B^R\psi^R = 0$
4. Determine the spacetime solution — $Y_{a\dot{a}} = (\psi^L_a)^t \psi^R_{\dot{a}}$
5. Read off the spacetime cross ratios
6. Compute the area — $\text{Area} = \int d^2z\, e^{\alpha}$

- The mathematics of this problem turns out to be the same as that of a problem studied by

D. Gaiotto, G. Moore & A. Neitzke - arXiv:0807.4723
- to appear

in the study of BPS states of 4d N=2 theories and wall crossing

↓

- 4d field theories on a circle ~ 3d field theory
- Hyperkahler metric on the Coulomb branch
- This metric contains the information we want

Moduli space of vacua of three dimensional field theories

- D4 brane (4+1 dimensional Yang Mills theory) on a Riemman surface → 2+1 dimensional field theory

Cherkis Kapustin
Gaiotto, Moore, Neitzke

$$D_{\bar{z}}\Phi_z = 0 \ , \qquad F_{z\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0$$

$$p = \mathrm{Tr}[\ \Phi_z^2]$$

Hitchin equations,
SU(2) group

Vacuua → solutions of the equations.

Moduli → coefficients in p(z).

Metric in moduli space → hyperkahler.

The same mathematical problem, we can use those results !!

U(1) theory + 1 hyper

Metric is known for the simplest case → $$\mathrm{p} = \mathrm{z}^2 - m$$

$$g_{m\bar{m}} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{|m|^2 + (n+1/2)^2}}$$

Ooguri-Vafa
Seiberg-Shenker

The area can be computed:

$$\text{Area} \sim m\partial_m K \ , \quad \longleftarrow \quad \partial_m \partial_{\bar{m}} K = g_{m\bar{m}}$$

In general: The moduli space has a U(1) symmetry under rotations in the plane.
The Area is the D term, or moment map, for the U(1) symmetry.

Regularizing the area

- Physical cutoff

$$r > \mu_{IR} \,, \qquad\qquad ds^2 = \frac{dx^+ dx^- + dr^2}{r^2}$$

- Need to know the solution.
- The same solution determines the position of the cusps
- → Write the answer in terms of the position of the cusps.

R = Remainder function

Usual one loop

$$A = A_{div} + A_{BDS} + A_{extra} + A^{reg}_{Kahler}$$

Usual divergent term

Previous formula with the infinity subtracted

Extra term which arises due to the regularization and depends on a certain ``magnetic'' cross ratio

Gaiotto, Moore and Neitzke have computed it

- Parameters of the polynomial in terms of the cross ratios

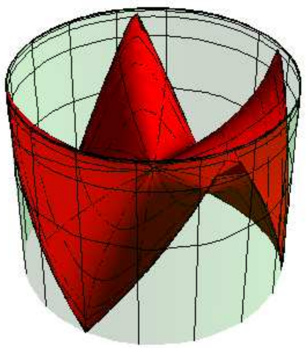
$$e^{\mathrm{Re}(m)} = \frac{x^+_{43} x^+_{21}}{x^+_{41} x^+_{32}} \qquad e^{\mathrm{Im}(m)} = \frac{x^-_{43} x^-_{21}}{x^-_{41} x^-_{32}}$$

Final answer for the octagon

$$R = \log\cosh \mathrm{Re}(m) \log\cosh \mathrm{Im}(m) + \frac{7\pi}{6} + \int dt \frac{(\bar{m}e^{t} - me^{-t})}{\tanh 2t} \log\left(1 - e^{-me^{t} + me^{-t}}\right)$$

$$e^{\mathrm{Re}(m)} = \frac{x^{+}_{43} x^{+}_{21}}{x^{+}_{41} x^{+}_{32}} \qquad e^{\mathrm{Im}(m)} = \frac{x^{-}_{43} x^{-}_{21}}{x^{-}_{41} x^{-}_{32}}$$

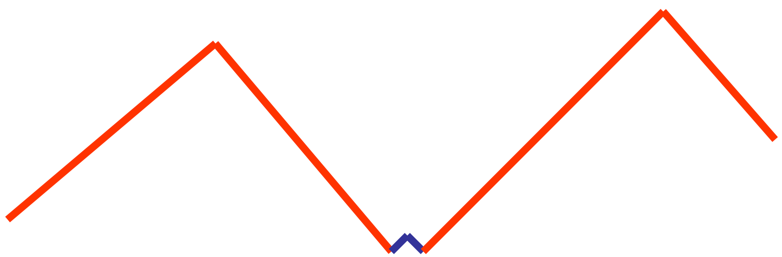
We did not need to find the explicit solution !



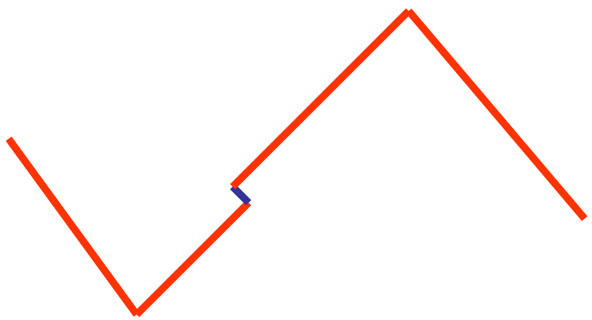
- R goes to a constant as m → Infinity

This constant is related to the solution for the hexagon.

- When m→ Infinity the cross ratios take extreme values and this corresponds to a a double soft or a soft-collinear limit



m→ infinity in a generic direction



m → infinity along a Stokes line.

Integrability and the general case

- Gaoitto Moore and Neitzke write down a system of equations that should determine the answer.
- They introduce a spectral parameter and consider the cross ratios as a function of the coefficients of the polynomial and the spectral parameter.
- The problem then displays a Stokes phenomenon in the spectral parameter, as it gets small or large.

Recipe

Choose p

$$p = z^{n-2} + m_{n-4} z^{n-4} + \cdots m_0$$

Introduce spectral parameter and think about cross ratios a functions of spectral parameter

$$\chi_i(\zeta; m_i, \bar{m}_i)$$

Write consistency conditions for the discontinuities of these cross ratios as funtions of ς

$$\chi_i \to \chi_i(1 - \chi_j)$$

Metric in moduli space

$$g_{m_i \bar{m}_j}$$

Compute the area

Conclusions

- We discussed amplitudes at strong coupling in N=4 SYM
- Dual conformal symmetry
- Relation to Wilson loops
- These symmetries fix the 4 and 5 gluon amplitudes
- For more gluons they leave an undetermined ``remainder'' function.

- This undetermined remainder function was computed at 2 loops and n=6

2 loops: Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich / Drummond, Henn, Korchemsky, Sokatchev. Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini

- We computed it at strong coupling and for 8 gluons.

- Connection to moduli spaces of three dimensional theories and to the wall crossing phenomenon.

- Hopefully one would understand how to do it for all values of the coupling.

Future

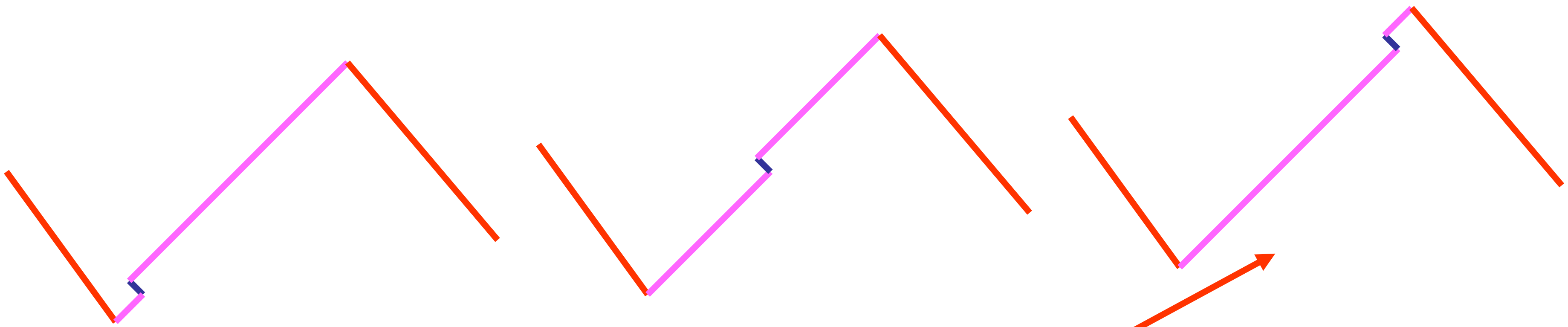
- Do the same for minimal surfaces in AdS_5

- Use these classical equations to understand the problem of operators and correlation functions.

- Generalize to all values of the coupling.

Ignore following slides

Wall Crossing



$$\log \epsilon^- \log \frac{\chi^+}{(1+\chi^+)}$$

$$\mathbf{A} \sim \log \epsilon^- \log \chi^+ \; .$$

$$A \sim -\log \epsilon^- \log \chi^+ + \log \chi^+ \log(1+\epsilon^-)$$

$$\chi^+ \to 0$$

$$\chi^+ \to \infty$$

$$|\log \epsilon^-| \gg |\log \chi^+|$$

Change in the coefficients of the subleading terms in the soft expansion. This change is the same at weak and strong coupling. The full coefficients are different at weak and strong coupling.

Recipe:

1- Start with p

2- Solve Sinh Gordon equation on the plane

3- Solve the linear problems on the plane

4- Construct:

$$Y_{a\dot{a}} = (\psi_a^L)^t \psi_{\dot{a}}^R$$

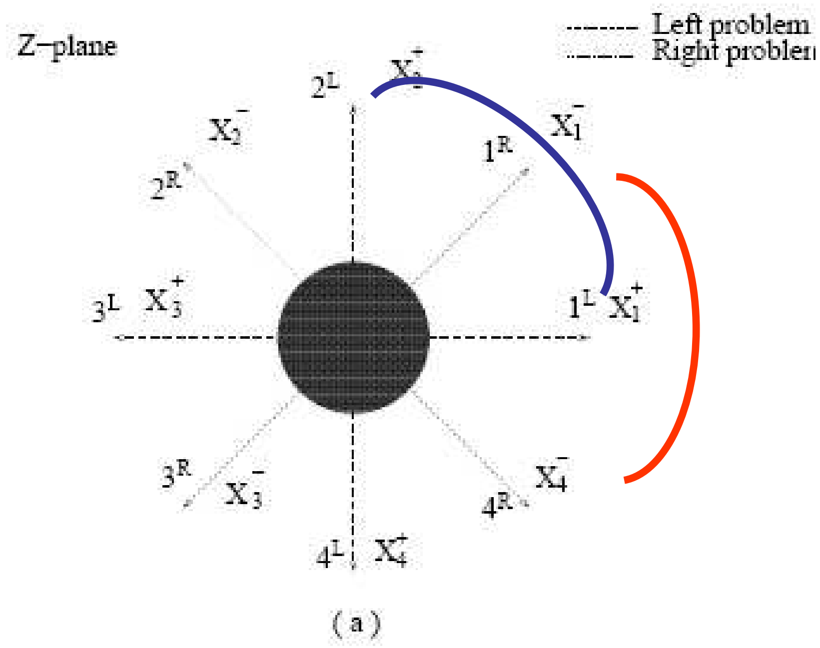
Solve equations on the whole complex plane

p is polynomial

$$p = z^{n-2} + \cdots$$

coefficients → related to spacetime cross ratios

Stokes Sectors



Solutions diverge in
different ways in different sectors

When we change sectors only left problem or only the right problem changes → Cusps are lightlike separated on the boundary.