Quantum Entropy Function

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Related work:
Murthy, Pioline

Introduction

The Bekenstein-Hawking formula gives a simple expression for the entropy of a black hole in a classical 2-derivative theory of gravity:

$$S_{BH}(Q) = A/4G_N$$

Wald's formula gives a generalization of this in a classical theory of gravity with arbitrary higher derivative interactions.

Is there a generalization of this formula in the full quantum theory of gravity that will count the exact degeneracies of black holes?

The best chance of finding such a formula is for extremal (supersymmetric) black holes.

After all, in the dual microscopic thory one can often count precisely the ground state degeneracy / index.

What macroscopic quantity should we compare it with?

Quantum entropy function is a proposal for such a macroscopic formula.

We shall work in some fixed duality frame and focus on single centered black holes.

In general the macroscopic degeneracy, denoted by d_{macro} can have two kinds of contributions:

1. From degrees of freedom living outside the horizon (hair)

Example: The fermion zero modes associated with the broken supersymmetry generators.

2. From degrees of freedom living inside the horizon.

We shall denote the degeneracy associated with these degrees of freedom by d_{hor}.

Our main goal: Find a macroscopic formula for d_{hor}.

Proposal for d_{hor}

Near horizon geometry of an extremal black hole always has the form of $AdS_2 \times K$.

K: some compact space, possibly fibered over AdS_2.

K includes the compact part of the space time as well as the angular coordinates in the black hole background, *e.g.* S^2 for a four dimensional black hole.

The near horizon geometry is separated from the asymptotic region by an infinite throat and is, by itself, a solution to the equations of motion.

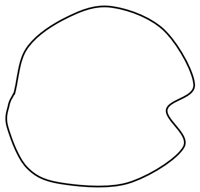
Thus we we expect d_{hor} to be given by some computation in the near horizon $AdS_2 \times K$ geometry.

Go to the euclidean formalism and represent the AdS_2 factor by the metric:

$$ds^2 = v\left((r^2-1)d\theta^2 + \frac{dr^2}{r^2-1}\right), \quad 1 \le r < \infty, \quad \theta \equiv \theta + 2\pi$$

We need to regularize the infinite volume of AdS_2 by putting a cut-off $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

$z = \sqrt{r^2-1}\, e^{i\theta}$ plane:



Proposal for d_{hor} (Quantum entropy function):

$$d_{hor} = Z^{(finite)}$$

$$Z = \left\langle \exp[-i q_k \oint d\theta \, A_\theta^{(k)}] \right\rangle$$

$\langle\ \rangle$: Path integral over string fields in the euclidean near horizon background geometry.

$\{q_k\}$: electric charges carried by the black hole, representing electric flux of the U(1) gauge field $A^{(k)}$ through AdS_2

$\oint$: integration along the boundary of AdS_2

finite: Infrared finite part of the amplitude.

Managing the infrared divergence:

Cut-off: $r \le r_0 f(\theta)$ for some smooth periodic function $f(\theta)$.

$\Rightarrow$ the boundary of AdS_2 has finite length $L \propto r_0$.

$Z^{(finite)}$ is defined by expressing Z as

$$Z = e^{CL+O(L^{-1})} \times Z^{(finite)}$$

C: A constant

Equivalently: $\ln Z^{(finite)} = \lim_{L\to\infty} \left(1 - L\frac{d}{dL}\right) \ln Z$

The definition can be shown to be independent of the choice of $f(\theta)$.

The role of

$$\exp\left[-i q_k \oint d\theta\, A_\theta^{(k)}\right].$$

In computing the path integral over AdS_2 we need to work in a fixed charge sector since the charge mode is non-normalizable and the mode associated with the chemical potential is normalizable.

$\Rightarrow$ **We need to add boundary terms in the action to make the path integral consistent.**

$\exp\left[-i q_k \oint d\theta\, A_\theta^{(k)}\right]$ **provides the required boundary term.**

Consistency checks:

1. In the classical limit

$$Z = \exp\left[-A_{bulk} - A_{boundary} - iq_k \oint d\theta\, A_\theta^{(k)}\right]$$

evaluated on the attractor geometry.

After extracting the finite part one finds:

$$Z^{(finite)} = \exp(S_{wald})$$

S_{wald}: Wald entropy of the black hole.

2. By AdS/CFT correspondence $Z = Z_{CFT_1}$.

CFT_1: Quantum mechanics obtained by taking the infrared limit of the brane system describing the black hole.

Since typically this theory has a gap, the infrared limit consists of just the ground states in a fixed charge sector.

$$\Rightarrow Z = d(q)\, e^{-E_0 L}$$

$(E_0, d(q))$: ground state (energy, degeneracy)

Thus $Z^{(finite)} = d(q)$.

Note: $d_{hor} = Z^{(finite)}$ computes the degeneracy for fixed charges, including angular momentum.

Thus this approach always gives the macroscopic entropy in the microcanonical ensemble.

An aside:

If we replace $AdS_2 \times K$ by $AdS_2 \times S^2$ and string theory by a superconformal gauge theory then the same expression computes the expectation value of a circular 't Hooft - Wilson loop operator.

Kapustin; Kapustin, Witten; Gomis, Okuda, Trancanelli

Degeneracy vs index

On the microscopic side we usually compute an index

On the other hand d_{hor} computes degeneracy.

How do we compare the two?

Strategy: Use d_{hor} to compute the index on the macroscopic side.

We shall illustrate this for the helicity trace B_n for a four dimensional single centered black hole.

For a black hole that breaks 2n supercharges we define

$$B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h} (2h)^n$$

h: 3rd component of angular momentum in rest frame

$$B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h_{hor}+2h_{hair}} (2h_{hor} + 2h_{hair})^n$$

In 4D only $h_{hor} = 0$ black holes are supersymmetric

$$\to B_n = (-1)^{n/2} \frac{1}{n!} \mathrm{Tr}(-1)^{2h_{hair}} (2h_{hair})^n = d_{hor}\, B_{n;hair}$$

If the only hair degrees of freedom are the fermion zero modes associated with the broken suspersymmetry generators then $B_{n;hair} = 1$, and hence $B_n = d_{hor}$.

Comparison with microscopic index

We shall consider quarter BPS dyons in type IIB string theory on K3 $\times$ S^1 $\times$ $\widetilde{S}^1$ and focus on a special class of states containing

D5/D3/D1 branes wrapped on 4/2/0 cycles of K3 $\times$ (S^1 or $\widetilde{S}^1$)

Q: D-brane charges wrapped on 4/2/0 cycles of K3 $\times$ $\widetilde{S}^1$

P: D-brane charges wrapped on 4/2/0 cycles of K3 $\times$ S^1

Q and P are each 24 dimensional vectors.

We shall try to explain some features of the microscopic index of this system using the quantum entropy function.

The relevant index is $B_6(Q, P)$ – the 6th helicity trace index of quarter BPS states carrying charges (Q, P).

Besides depending on the charges, $B_6(Q, P)$ also depends on the asymptotic values of the moduli fields as the degeneracy can jump as we cross walls of marginal stability.

In order to facilitate comparison with the macroscopic results we shall choose the asymptotic moduli such that only single centered black holes contribute to $B_6(Q, P)$.

Duality symmetries

The duality symmetries which take D-branes to D-branes is given by

$$O(4,20;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$$

An arithmetic invariant of $O(4,20;\mathbb{Z})_T \times SL(2,\mathbb{Z})_S$:

$$\ell \equiv \gcd\{Q_iP_j - Q_jP_i\}$$

Dabholkar, Gaiotto, Nampuri

With the help of $SL(2,\mathbb{Z})_S$ transformation any charge vector can be brought to the form

$$(Q,P) = (\ell Q_0, P_0), \quad \gcd\{Q_{0i}P_{0j} - P_{0i}Q_{0j}\} = 1$$

Banerjee, A.S.

We shall proceed with this choice.

Intersection form of 4/2/0 forms on $K3$ defines additional $O(4,20;\mathbb{Z})$ invariants

$$Q^2, \qquad P^2, \qquad Q\cdot P$$

One finds that for $(Q,P) = (\ell Q_0, P_0)$ the microscopic result for $B_6(Q,P)$ takes the form

$$\sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s), \qquad s|\ell \Leftrightarrow \ell/s \in \mathbb{Z}$$

Banerjee, A.S., Srivastava; Dabholkar, Gomes, Murthy

$f(m,n,p)$: Fourier transform of the inverse of Igusa cusp form

$$B_6(Q,P) = \sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q\cdot P/s)$$

Note: For $\ell = 1$ only the $s = 1$ terms contribute.

Dijkgraaf, Verlinde, Verlinde

Our goal will be to try to understand the extra terms for $\ell > 1$ from the macroscopic viewpoint.

For large charges

$$\begin{aligned} & f(Q^2/s^2, P^2, Q\cdot P/s) \\ = \quad & \exp\left[\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right] \\ & \times \text{ Series expansion in inverse powers of charges} \\ & + \text{Exponentially suppressed corrections} \end{aligned}$$

$$B_6(Q, P) = \sum_{s|\ell} s\, f(Q^2/s^2, P^2, Q \cdot P/s)$$

Note: $s = 1$ term always contributes.

For large charges this gives the leading contribution

$$\exp\left[\pi\sqrt{Q^2P^2 - (Q \cdot P)^2}\right] = \exp\left[S_{wald}\right]$$

Our goal: Understand the extra terms which appear in the microscopic formula for $\ell > 1$ from the macroscopic viewpoint.

Strategy: Look for additional saddle points in the path integral with the following properties:

1. It must be parametrized by an integer s satisfying the constraint $\ell/s \in \mathbb{Z}$

2. The classical contribution to $Z^{(\mathrm{finite})}$ from this saddle point must be equal to

$$\exp\left[\pi\sqrt{Q^2P^2-(Q\cdot P)^2}/s\right]$$

3. It must preserve sufficient amount of supersymmetry so as not to vanish due to fermion zero mode integration.

There is indeed such saddle points in the path integral, constructed as follows.

1. Take the original near horizon geometry of the black hole.

2. Take a $\mathbb{Z}_s$ orbifold of this background with $\mathbb{Z}_s$ acting as

a) $2\pi/s$ rotation in AdS_2

a) $2\pi/s$ rotation in S^2

c) $2\pi/s$ unit of translation along the circle S^1.

– freely acting $\mathbb{Z}_s$.

1. Quantization of the RR 3-form flux requires that $\ell/s \in \mathbb{Z}$.

2. Ordinarily an orbifold of this type will change the asymptotic structure of space time but in AdS_2 it preserves the asymptotic boundary conditions.

3. The contribution to Z from this saddle point is given by

$$e^{CL} \exp\left[\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right]$$

Thus its contribution to $Z^{(finite)}$ is of order

$$\exp\left[\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right]$$

Furthermore these saddle points preserve sufficient amount of supersymmeries so that integration over the fermion zero modes associated with the broken supersymmetries do not make the path integral vanish automatically.

Banerjee, Banerjee, Gupta, Mandal, A.S.

Orbifold action:

$$\theta \to \theta + 2\pi/s, \quad \phi \to \phi + 2\pi/s, \quad x^5 \to x^5 + 2\pi/s$$

At AdS_2 center ($r = 1$) the shift in θ is irrelevant.

$\to$ **the identification is** $(\phi, x^5) \equiv (\phi + 2\pi/s, x^5 + 2\pi/s)$.

Thus the RR flux Q through the cycle at $r = 1$, spanned by (x^5, ψ, ϕ) gets divided by s.

Flux quantization $\to$ the orbifold is well defined only if Q is divisible by s, i.e. if

$$\ell/s \in \mathbb{Z}$$

Denoting the (r,θ,ϕ,x^5) coordinates of the orbifold by $(\widetilde{r},\widetilde{\theta},\widetilde{\phi},\widetilde{x}^5)$ we get the new metric

$$ds^2 = v\left[\frac{d\widetilde{r}^2}{\widetilde{r}^2-1} + (\widetilde{r}^2-1)d\widetilde{\theta}^2\right] + w\left[d\psi^2 + \sin^2\psi\, d\widetilde{\phi}^2\right] + \cdots$$

$$(\widetilde{\theta}+2\pi/s, \widetilde{\phi}+2\pi/s, \widetilde{x}^5+2\pi/s) \equiv (\widetilde{\theta},\widetilde{\phi},\widetilde{x}^5)$$

Define

$$\theta = s\widetilde{\theta}, \quad r = \widetilde{r}/s, \quad \phi = \widetilde{\phi}-\widetilde{\theta}, \quad x^5 = \widetilde{x}^5 - \widetilde{\theta}$$

Then

$$ds^2 = v\left(\frac{dr^2}{r^2-s^{-2}} + (r^2-s^{-2})\,d\theta^2\right) + w[d\psi^2 + \sin^2\psi(d\phi + s^{-1}d\theta)^2]$$

$$(\theta+2\pi, \phi, x^5) \equiv (\theta,\phi,x^5)$$

Its contribution to $d_{hor}(Q, P)$ in the classical limit is given by

$$\exp[S_{wald}/s] = \exp\left[2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}/s\right]$$

This is the same behaviour as of $f(Q^2/s^2, P^2, Q\cdot P/s)$.

Note: The infrared divergent part is also divided by s and gives a contribution to the exponent:

$$C\,\widetilde{L}/s = C\,L + O(L^{-1})$$

Final comments

In principle one should be able to reproduce the full macroscopic partition function from path integral over the near horizon $AdS_2 \times K$ geometry.

This would seem to be difficult task as it involves path integral over all the string fields.

However one can argue that supersymmetry restricts the path integral to over configurations preserving a certain amount of supersymmetry.

Beasley, Gaiotto, Guica, Huang, Strominger, Yin

Banerjee, Banerjee, Gupta, Mandal, A.S.

Hope: Using this result one can collapse the whole path integral to a finite dimensional integral which can then be computed.

In this context it is amusing to note that even the microscopic degeneracy formula in this theory can be expressed as a sum of contributions over different saddle points, with the contribution from each saddle point being given by a two dimensional integral.

Once we are confident that the formalism works for N=4 black holes, we can then use it to compute the degeneracies of N=2 black holes where the microscopic formula is still unknown.

Reissner-Nordstrom solution in $D = 4$:

$$
\begin{aligned}
ds^2 &= -(1-\rho_+/\rho)(1-\rho_-/\rho)d\tau^2 \\
&\quad + \frac{d\rho^2}{(1-\rho_+/\rho)(1-\rho_-/\rho)} \\
&\quad + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

Define

$$
2\lambda = \rho_+ - \rho_-, \quad t = \frac{\lambda\tau}{\rho_+^2}, \quad r = \frac{2\rho - \rho_+ - \rho_-}{2\lambda}
$$

and take $\lambda \to 0$ limit.

$$
ds^2 = \rho_+^2\left[-(r^2-1)dt^2 + \frac{dr^2}{r^2-1}\right] + \rho_+^2(d\theta^2 + \sin^2\theta d\phi^2)
$$