$D = 3$ Duality and Black Holes

K.S. Stelle

Imperial College London

Strings 2009 Conference

Rome, 25 June 2009

G. Bossard, H. Nicolai & K.S.S. 0809.5218 and 0902.4438
G. Bossard and H. Nicolai 0906.1986
G. Bossard 0906.1988

Some questions in classical supergravity

(Also see talk by S. Ferrara)

- Can one use duality symmetries to relate non-extremal and extremal solutions?
- How do large duality symmetries like E_8 act on small parameter spaces such as one with just 58 moduli?
- What happens when gravitational parameters such as mass and NUT charge are included among the duality moduli?
- What is the relationship between extremal and BPS solutions?
- Should there be $D = 3$ duality group discretization?

Answers will involve the following concepts

- Timelike dimensional reduction
- Scalar field Noether charges
- The rôles of parabolic subgroups
- BPS Geology and Iwasawa failure sets
- Nilpotent duality orbits $\Longrightarrow$ extremal orbit classification
- The BPS "Dirac equation"

Stationary solutions and timelike dimensional reduction

The search for supergravity solutions with assumed Killing symmetries can be recast as a Kaluza-Klein problem. Consider a $D = 4$ theory with a nonlinear bosonic symmetry $\bar{G}$ (*e.g.* E_7 for maximal $N = 8$ supergravity). Scalar fields take their values in a target space $\bar{\Phi} = \bar{G}/\bar{H}$, where $\bar{H}$ is the corresponding linearly realized subgroup, generally the maximal compact subgroup of $\bar{G}$ (*e.g.* $\mathrm{SU}(8)$ for $N = 8$ SG).

Searching for stationary solutions to such a theory amounts to assuming further that a solution possesses a timelike Killing vector field $\kappa_\mu(x)$.

- We assume that the solution spacetime is asymptotically flat or asymptotically Taub-NUT and that there is a 'radial' function r which is divergent in the asymptotic region, $g^{\mu\nu}\partial_\mu r \partial_\nu r \sim 1 + \mathcal{O}(r^{-1})$.
- The Killing vector κ will be assumed to have $W := -g_{\mu\nu}\kappa^\mu\kappa^\nu \sim 1 + \mathcal{O}(r^{-1})$.

- We assume asymptotic hypersurface orthogonality, $\kappa^{\nu}(\partial_{\mu}\kappa_{\nu} - \partial_{\nu}\kappa_{\mu}) \sim \mathcal{O}(r^{-2})$.
- In any vielbein frame, the curvature will fall off as $R_{abcd} \sim \mathcal{O}(r^{-3})$.
- Lie derivatives with respect to κ are assumed to vanish on all fields.

The $D = 3$ theory dimensionally reduced with respect to a timelike Killing vector κ will have an Abelian principal bundle structure, with a metric

$$ds^2 = -W(dt + \hat{B}_i dx^i)^2 + W^{-1}\gamma_{ij} dx^i dx^j$$

where t is a coordinate adapted to the Killing vector κ and γ is the metric on the 3-dimensional hypersurface Σ_3 at constant t. If the $D = 4$ theory also has Abelian vector fields $\mathcal{A}_{\mu}$, they similarly reduce to $D = 3$ as

$$4\sqrt{4\pi G}\mathcal{A}_{\mu} dx^{\mu} = U(dt + \hat{B}_i dx^i) + \hat{A}_i dx^i$$

Comparison to spacelike dimensional reductions

The timelike $D = 3$ reduced theory will have a G/H^* coset space structure similar to the G/H coset space structure of a $D = 3$ theory reduced with a *spacelike* Killing vector. Thus, for a spacelike reduction of maximal supergravity, one obtains an $\mathrm{E}_8/\mathrm{SO}(16)$ theory continuing on in the sequence of dimensional reductions descending from $D = 11$. Julia Similarly to the analogous spacelike reduction, the timelike-reduced $D = 3$ theory has the possibility of exchanging $D = 3$ Abelian vector fields for scalars by dualization, contributing to the appearance of an enlarged $D = 3$ bosonic duality symmetry. The resulting $D = 3$ theory then contains $D = 3$ gravity coupled to a G/H^* nonlinear sigma model.

- However, although the numerator group G is the same as that obtained in a spacelike reduction, the divisor group H^* for a timelike reduction is a *noncompact* form of the spacelike divisor group H. Breitenlohner, Gibbons & Maison 1988
- A consequence of this $H \to H^*$ change and the dualization of vectors is the appearance of *negative-sign* kinetic terms for some $D = 3$ scalars.

Example: maximal supergravity in $D = 3$

Maximal supergravity after a timelike reduction to $D = 3$ and subsequent dualization of 29 vectors to scalars has a bosonic sector containing $D = 3$ gravity coupled to a $E_8/\mathrm{SO}^*(16)$ nonlinear sigma model with 128 scalar fields. As a consequence of the timelike dimensional reduction and vector dualizations, however, the scalars do not all have the same signs for their "kinetic" terms.

• There are 72 positive-sign scalars: 70 descending directly from the $D = 4$ theory, one emerging from the $D = 4$ metric and one more coming from the $D = 4 \to D = 3$ Kaluza-Klein vector, subsequently dualized to a scalar.

• There are 56 negative-sign scalars: 28 descending directly from the time components of the 28 $D = 4$ vectors, and another 28 emerging from the $D = 3$ vectors obtained from spatial components of the 28 $D = 4$ vectors, becoming then negative-sign scalars after dualization.

The sigma model structure of this timelike reduced theory is $E_8/\mathrm{SO}^*(16)$. The $\mathrm{SO}^*(16)$ divisor group is not an $\mathrm{SO}(p,q)$ group defined via preservation of an indefinite metric. Instead it is constructed starting from the $\mathrm{SO}(16)$ Clifford algebra $\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}$ and then by forming the complex $U(8)$-covariant oscillators $a_i := \frac{1}{2}(\Gamma_{2i-1} + i\Gamma_{2i})$ and $a^i \equiv (a_i)^\dagger = \frac{1}{2}(\Gamma_{2i-1} - i\Gamma_{2i})$. These satisfy the standard fermi oscillator creation/annihilation anticommutation relations

$$\{a_i, a_j\} = \{a^i, a^j\} = 0 \quad , \qquad \{a_i, a^j\} = \delta_i{}^j$$

The 120 $\mathrm{SO}^*(16)$ generators are then formed from the 64 hermitean $U(8)$ generators $a_i{}^j$ plus the $2 \times 28 = 56$ *antihermitean* combinations of $a_{ij} \pm a^{ij}$.

Under $\mathrm{SO}^*(16)$, the vector representation and the antichiral spinor are pseudo-real, while the 128-dimensional chiral spinor representation is *real*. This is the representation under which the 72+56 scalar fields transform in the $E_8/\mathrm{SO}^*(16)$ sigma model.

Some examples of G/H and G/H^* theories in $D = 3$

G/H	G/H^*	$\bar{G}/\bar{H}$	3+1 dimensional theory
$\frac{SL(n+2)}{SO(n+2)}$	$\frac{SL(n+2)}{SO(n,2)}$	$GL(n)/SO(n)$	$n+4$ dimensional Einstein gravity with n Killing vectors
$\frac{SU(2,1)}{S(U(2)\times U(1))}$	$\frac{SU(2,1)}{S(U(1,1)\times U(1))}$	$U(1)/U(1)$	Einstein-Maxwell ($N=2$ supergravity)
$\frac{SO(8,2)}{SO(8)\times SO(2)}$	$\frac{SO(8,2)}{SO(6,2)\times SO(2)}$	$\frac{SO(6)\times SO(2,1)}{SO(6)\times SO(2)}$	$N=4$ supergravity
$\frac{SO(8,8)}{SO(8)\times SO(8)}$	$\frac{SO(8,8)}{SO(6,2)\times SO(2,6)}$	$\frac{SO(6,6)\times SO(2,1)}{SO(6)\times SO(6)\times SO(2)}$	$N=4$ supergravity +supersym. Maxwell (10 dim. supergravity)
$E_{8(+8)}/SO(16)$	$E_{8(+8)}/SO^*(16)$	$E_{7(+7)}/SU(8)$	$N=8$ supergravity (11 dim. supergravity)

The $D = 3$ classification of extended supergravity stationary solutions via timelike reduction generalizes the $D = 3$ supergravity systems obtained from spacelike reduction. de Wit, Tollsten & Nicolai . This also connects with $N = 2$ models with coupled vectors Meesen & Ortin and $N = 4$ models with vectors where solutions have also been generated using duality symmetries. Cvetic & Youm, Cvetic & Tseytlin

Charges

Define the Komar two-form $K \equiv \partial_\mu \kappa_\nu dx^\mu \wedge dx^\nu$. This is invariant under the action of the timelike isometry and, by the asymptotic hypersurface orthogonality assumption, is asymptotically horizontal. This condition is equivalent to the requirement that the scalar field B dual to the Kaluza-Klein vector arising out of the $D = 4$ metric must vanish like $\mathcal{O}(r^{-1})$ as $r \to \infty$. In this case, one can define the Komar mass and NUT charge by (where s^* indicates a pull-back to a section) Bossard, Nikolai & K.S.S.

$$m \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* \star K \qquad n \equiv \frac{1}{8\pi} \int_{\partial\Sigma} s^* K$$

The Maxwell field also defines charges. Using the Maxwell field equation $d \star \mathcal{F} = 0$, where $\mathcal{F} \equiv \delta\mathcal{L}/\delta F$ is a linear combination of the two-form field strengths F depending on the $D = 4$ scalar fields, and using the Bianchi identity $dF = 0$, one obtains conserved electric and magnetic charges:

$$q \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* \star \mathcal{F} \qquad p \equiv \frac{1}{2\pi} \int_{\partial\Sigma} s^* F$$

Now consider these charges from the three-dimensional point of view in order to clarify their transformation properties under the $D = 3$ duality group G (in a simple Maxwell-Einstein example, one has simply $G = \mathrm{SU}(2,1)$).

The three-dimensional theory is described in terms of a coset representative $\mathcal{V} \in G/H^*$. The Maurer–Cartan form $\mathcal{V}^{-1} d\mathcal{V}$ decomposes as

$$\mathcal{V}^{-1} d\mathcal{V} = Q + P \quad , \qquad Q \equiv Q_\mu dx^\mu \in \mathfrak{h}^* \quad , \quad P \equiv P_\mu dx^\mu \in \mathfrak{g} \ominus \mathfrak{h}^*$$

Then the three-dimensional equations of motion can be rewritten as $d \star \mathcal{V} P \mathcal{V}^{-1} = 0$, so the $\mathfrak{g}$-valued Noether current is $\star \mathcal{V} P \mathcal{V}^{-1}$.

Since the three-dimensional theory is Euclidean, one cannot properly speak of a conserved charge. Nevertheless, since $\star \mathcal{V} P \mathcal{V}^{-1}$ is d-closed, the integral of this 2-form over a given homology cycle does not depend on the particular representative of that cycle.

As a result, for stationary solutions, the integral of this three-dimensional current, over any space-like closed surface containing in its interior all the singularities and topologically non-trivial subspaces of a solution, defines a $\mathfrak{g} \ominus \mathfrak{h}^*$-valued charge matrix $\mathscr{C}$

$$\mathscr{C} \equiv \frac{1}{4\pi} \int_{\partial\Sigma} \star \mathcal{V} P \mathcal{V}^{-1}$$

This transforms in the adjoint representation of G according to the standard non-linear action. For asymptotically flat solutions, $\mathcal{V}$ can be arranged to tend to the identity matrix asymptotically and the charge matrix $\mathscr{C}$ in that case is given by the asymptotic value of the one-form P:

$$P = \mathscr{C} \, \frac{dr}{r^2} + \mathcal{O}(r^{-2})$$

Now set up some general notation for the relevant group structure.

Let $\mathfrak{g}_4$ be the algebra of the $D = 4$ symmetry group $\bar{G}$ and let $\mathfrak{h}_4$ be the algebra of its $D = 4$ divisor group $\bar{H}$. $\mathfrak{sl}(2, \mathbb{R}) \cong \mathfrak{so}(2, 1)$ is the algebra of the Ehlers group (*i.e.* the $D = 3$ duality group of pure $D = 4$ gravity); $\mathfrak{so}(2)$ is the algebra of its divisor group. Let $\mathfrak{l}_4$ be the $\mathfrak{h}_4$ representation carried by the electric and magnetic charges q and p. Then $\mathscr{C}$ can be decomposed into three irreducible representations with respect to $\mathfrak{so}(2) \oplus \mathfrak{h}_4$ according to

$$\mathfrak{g} \ominus \mathfrak{h}^* \cong \big(\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)\big) \oplus \mathfrak{l}_4 \oplus \big(\mathfrak{g}_4 \ominus \mathfrak{h}_4\big)$$

The metric induced by the $\mathfrak{g}$ algebra's Cartan-Killing metric on this coset space is positive definite for the first and last terms, but is negative definite for $\mathfrak{l}_4$.

One associates the $\mathfrak{sl}(2, \mathbb{R}) \ominus \mathfrak{so}(2)$ component above with the Komar mass and the Komar NUT charge, and one associates the $\mathfrak{l}_4$ components with the electromagnetic charges. The remaining $\mathfrak{g}_4 \ominus \mathfrak{h}_4$ charges come from the Noether current of the original four-dimensional theory.

Characteristic equation

Breitenlohner, Gibbons and Maison proved that if G is simple, all the non-extremal single-black-hole solutions of a given theory lie on the H^* orbit of a Kerr solution. Moreover, all *static* solutions regular outside the horizon with a charge matrix satisfying $\mathrm{Tr}\,\mathscr{C}^2 > 0$ lie on the H^*-orbit of a Schwarzschild solution. (Turning on and off angular momentum requires consideration of the $D = 2$ duality group generalizing the Geroch A_1^1 group.)

Using Weyl coordinates, the coset representative $\mathcal{V}$ associated to the Schwarzschild solution with mass m can be written in terms of the non-compact generator $\mathbf{h}$ of the Ehlers $\mathfrak{sl}(2,\mathbb{R})$ only, *i.e.*

$$\mathcal{V} = \exp\left(\frac{1}{2}\ln\frac{r-m}{r+m}\,\mathbf{h}\right) \quad \rightarrow \qquad \mathscr{C} = m\mathbf{h}$$

For the maximal $N = 8$ theory with symmetry $\mathrm{E}_{8(8)}$ (and also for the exceptional 'magic' $N = 2$ supergravity Gunaydin, Sierra & Townsend with symmetry $\mathrm{E}_{8(-24)}$), one finds

$$\mathbf{h}^5 = 5\mathbf{h}^3 - 4\mathbf{h}$$

- Consequently, the charge matrix $\mathscr{C}$ satisfies in all cases

$$\boxed{\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}}$$

  where $c^2 \equiv \frac{1}{k} \operatorname{Tr} \mathscr{C}^2$ is the extremality parameter (vanishing for extremal static solutions) and $k \equiv \operatorname{Tr} \mathbf{h}^2 > 0$.
- Moreover, for all but the two exceptional E_8 cases, a stronger constraint is actually satisfied by the charge matrix $\mathscr{C}$:

$$\boxed{\mathscr{C}^3 = c^2\mathscr{C}}$$

The characteristic equations select acceptable orbits of solutions, *i.e.* orbits not exclusively containing solutions with naked singularities. They determine $\mathscr{C}$ in terms of the mass and NUT charge and the $D = 4$ electromagnetic charges.

Supersymmetry 'Dirac equation'

Extremal solutions have $c^2 = 0$, implying that the charge matrix $\mathscr{C}$ becomes *nilpotent*: $\mathscr{C}^5 = 0$ in the E_8 cases and $\mathscr{C}^3 = 0$ otherwise.

For $\mathcal{N}$ extended supergravity theories, one finds $H^* \cong \mathrm{Spin}^*(2\mathcal{N}) \times H_0$ and the charge matrix $\mathscr{C}$ transforms as a Weyl spinor of $\mathrm{Spin}^*(2\mathcal{N})$ valued in a representation of $\mathfrak{h}_0$. As in the $\mathrm{SO}^*(16)$ case we considered earlier, define the $\mathrm{Spin}^*(2\mathcal{N})$ fermionic oscillators

$$a_i := \frac{1}{2}\Big(\Gamma_{2i-1} + i\Gamma_{2i}\Big) \qquad a^i \equiv (a_i)^\dagger = \frac{1}{2}\Big(\Gamma_{2i-1} - i\Gamma_{2i}\Big)$$

for $i, j, \cdots = 1, \ldots, \mathcal{N}$. These obey standard creation & annihilation anticommutation relations.

Using this creation/annihilation oscillator basis, the charge matrix $\mathscr{C}$ can be represented as a state

$$|\mathscr{C}\rangle \equiv \Big(w + Z_{ij} a^i a^j + \Sigma_{ijkl} a^i a^j a^k a^l + \cdots\Big)\,|0\rangle$$

From the requirement that dilatino fields be left invariant under an unbroken supersymmetry of a BPS solution, one derives a 'Dirac equation' for the charge state vector,

$$\left(\epsilon^{i}_{\alpha} a_i + \Omega_{\alpha\beta}\epsilon^{\beta}_{i} a^i\right)|\mathscr{C}\rangle = 0$$

where $(\epsilon^i_\alpha, \epsilon^\alpha_i)$ is the asymptotic (for $r \to \infty$) value of the Killing spinor and $\Omega_{\alpha\beta}$ is a symplectic form on $\mathbb{C}^{2n}$ in cases with n/N preserved supersymmetry.

This condition turns out to be equivalent to the algebraic requirement that $\mathscr{C}$ be a *pure spinor* of $\mathrm{Spin}^*(2\mathcal{N})$ For BPS solutions, it has the consequence that the characteristic equations can be explicitly solved in terms of rational functions.

Note that $c^2 = 0 \Longleftrightarrow \langle\mathscr{C}|\mathscr{C}\rangle = 0$ is a *weaker* condition than the supersymmetry Dirac equation. Extremal and BPS are not always synonymous conditions, although they coincide for $\mathcal{N} \leq 5$ pure supergravities. They are not synonymous for $\mathcal{N} = 6\,\&\,8$ or for theories with vector matter coupling.

BPS Geology

Analysis of the 'Dirac equation' or nilpotency degree of the charge matrix $\mathscr{C}$ leads to a decomposition of the moduli space $\mathcal{M}$ of supergravity solutions into *strata* of various BPS degrees. Letting $\mathcal{M}_0$ be the non-BPS stratum, $\mathcal{M}_1$ being the $\frac{1}{N}$ BPS stratum, etc., the dimensions of some of the strata for pure supergravity theories turn out to be

	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=8$
$\dim(\mathcal{M}_0)$	4	8	14	22	34	58
$\dim(\mathcal{M}_1)$	3	7	13	21	33	57
$\dim(\mathcal{M}_2)$			8	16	26	46
$\dim(\mathcal{M}_4)$					17	29

Parabolic cosets

Where do such stratum dimensions come from? Take the non-extremal stratum of $N = 8$ supergravity as an example, with 58 moduli. In order for this small number to be related to an E_8 group action, one needs to find a *large* isotropy group to divide by.

The existence of such a large subgroup is a peculiarity of non-compact groups, analogous to the 4-generator Borel subgroup of the $D = 4$ Lorentz group . For the non-extremal $N = 8$ supergravity stratum $\mathcal{M}_0$, there is a 190 generator parabolic subgroup $\mathcal{P}_0$ containing the $D = 4$ duality group E_7, 56 generators corresponding to the 56 electromagnetic charges of the $D = 4$ theory, plus one more generator. The resulting $E_8/\mathcal{P}_0$ coset is then 58 dimensional. However, as we shall see, this gives a proper group action only on a dense subset of the full moduli space.

Analysis of the extremal strata of supergravity solutions requires understanding the *nilpotent orbits* of the $D = 3$ duality group G. This analysis links up with established mathematical literature on nilpotent orbits, in particular by Đoković.

‘Almost Iwasawa’ decomposition

Earlier analysis of the orbits of the $D = 4$ symmetry groups $\bar{G}$ Cremmer, Lü, Pope & K.S.S. heavily used the Iwasawa decomposition

$$g = u_{(g,Z)} \exp\Big(\ln \lambda_{(g,Z)} \mathbf{z} \Big) \, b_{(g,Z)}$$

with $u_{(g,Z)} \in \bar{H}$ and $b_{(g,Z)} \in \mathfrak{B}_Z$ where $\mathfrak{B}_Z \subset \bar{G}$ is the parabolic subgroup that leaves the charges Z invariant up to a multiplicative factor $\lambda_{(g,Z)}$. This multiplicative factor can be compensated for by ‘trombone’ transformations combining Weyl scalings with compensating dilational coordinate transformations, leading to a formulation of active symmetry transformations that map solutions into other solutions with *unchanged asymptotic values* of the spacetime metric and scalar fields.

- The $D = 3$ structure is characterized by the fact that the Iwasawa decomposition *breaks down* for noncompact divisor groups H^*.
- The Iwasawa decomposition does, however work “almost everywhere” in the $D = 3$ solution space. The places where it fails are precisely the extremal suborbits of the duality group.

Multicentered Solutions Bossard & Nicolai

The above framework applies equally to multi-centered as to single-centered solutions. One may start from a general ansatz

$$\mathcal{V}(x) = \mathcal{V}_0 \exp(-\sum_n \mathcal{H}^n(x)\mathscr{C}_n)$$

with Lie algebra elements $\mathscr{C}_n \in \mathfrak{g} \ominus \mathfrak{h}^*$ and functions $\mathcal{H}^n(x)$ to be determined by the equations of motion. Defining as above $\mathcal{V}^{-1}d\mathcal{V} = Q + P$ and restricting P to depend linearly on the functions $\mathcal{H}^n(x)$, one finds the requirement $[\mathscr{C}_m, [\mathscr{C}_n, \mathscr{C}_p]] = 0$. The Einstein and scalar equations of motion then reduce to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \sum_{mn} \partial_\mu \mathcal{H}^m \partial_\nu \mathcal{H}^n \,\mathrm{Tr}\, \mathscr{C}_m\mathscr{C}_n \qquad d \star d\mathcal{H}^n = 0$$

Restricting attention to solutions where the 3-space is flat then requires $\mathrm{Tr}\, \mathscr{C}_m\mathscr{C}_n = 0$. The resulting system generalizes that found by Clement & Galt'sov. Solving $[\mathscr{C}_m, [\mathscr{C}_n, \mathscr{C}_p]] = 0 = \mathrm{Tr}\, \mathscr{C}_m\mathscr{C}_n$ is now reduced to an algebraic problem amenable to the above orbit analysis: non-extremal and extremal stationary solutions can be formed from extremal single-hole constituents.

Arithmetic subgroups?

Since the work of Hull & Townsend, there has been a 'folk' expectation that all Cremmer-Julia type duality symmetries should be reduced to arithmetic subgroups like $E_8(\mathbb{Z})$ as a result of Dirac charge quantization. However, consider the explicit transformations of the pure gravity charge matrix

$$\mathscr{C} \equiv \begin{pmatrix} m & n \\ n & -m \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \ominus \mathfrak{so}(2)$$

yielding

$$m' = \frac{(\alpha^2 - \gamma^2 + \beta^2 - \delta^2)c + (\alpha^2 - \gamma^2 - \beta^2 + \delta^2)m + 2(\alpha\beta - \gamma\delta)n}{\sqrt{2(\alpha^2 + \gamma^2 + \beta^2 + \delta^2) + 2(\alpha^2 + \gamma^2 - \beta^2 - \delta^2)\frac{m}{c} + 4(\alpha\beta + \gamma\delta)}}$$
$$n' = \frac{2(\alpha\gamma + \beta\delta)c + 2(\alpha\gamma - \beta\delta)m + 2(\alpha\delta + \beta\gamma)n}{\sqrt{2(\alpha^2 + \gamma^2 + \beta^2 + \delta^2) + 2(\alpha^2 + \gamma^2 - \beta^2 - \delta^2)\frac{m}{c} + 4(\alpha\beta + \gamma\delta)}}$$

It is very hard to see how such transformations can be discretized in such a way as to preserve a Dirac type quantization rule.

Conclusions

What has been developed is a quite general framework for the analysis of stationary supergravity solutions using duality orbits.

- The Noether charge matrix $\mathscr{C}$ satisfies a characteristic equation $\mathscr{C}^5 = 5c^2\mathscr{C}^3 - 4c^4\mathscr{C}$ in the maximal E_8 cases and $\mathscr{C}^3 = c^2\mathscr{C}$ in the non-maximal cases, where $c^2 \equiv \frac{1}{k}\,\mathrm{Tr}\,\mathscr{C}^2$ is the extremality parameter.
- Extremal solutions are characterized by $c^2 = 0$, and $\mathscr{C}$ becomes nilpotent ($\mathscr{C}^5 = 0$ *viz.* $\mathscr{C}^3 = 0$) on the corresponding extremal suborbits.
- BPS solutions have a charge matrix $\mathscr{C}$ satisfying an algebraic 'supersymmetry Dirac equation' which encodes the general properties of such solutions. This is a stronger condition than the $c^2 = 0$ extremality condition.
- The orbits of the $D = 3$ duality group G are not always acted upon transitively by G. This is related to the failure of the Iwasawa decomposition for noncompact divisor groups H^*. The Iwasawa failure set corresponds to the extremal suborbits.